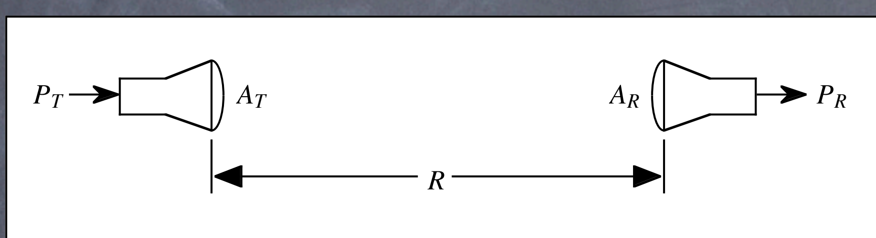
Session 3

3.1

The Friis Equation

- Suppose we have two antennas "pointing at each other" a large distance R apart.



If

$$P_T = \text{transmitted power}$$
$$P_R = \text{received power}$$

then

$$\frac{P_R}{P_T} = \frac{A_T A_R}{\lambda^2 R^2}. \quad \text{(Friis Equation)}$$

$$\frac{P_R}{P_T} = \frac{A_T A_R}{\lambda^2 R^2}$$

We must be in the "far field" and looking at small angles off "boresight":

$$R \gg \frac{D_{\text{max}}^2}{4\lambda}$$

The aperture must be large enough for scalar diffraction to be accurate:

$$D_{\text{min}} \gg \lambda$$

The Friis Equation

$$\frac{P_R}{P_T} = \frac{A_T A_R}{\lambda^2 R^2}$$

Intuitively

$$\frac{P_R}{P_T} = \left(\frac{A_T}{\lambda^2}\right) (A_R) \left(\frac{1}{R^2}\right)$$

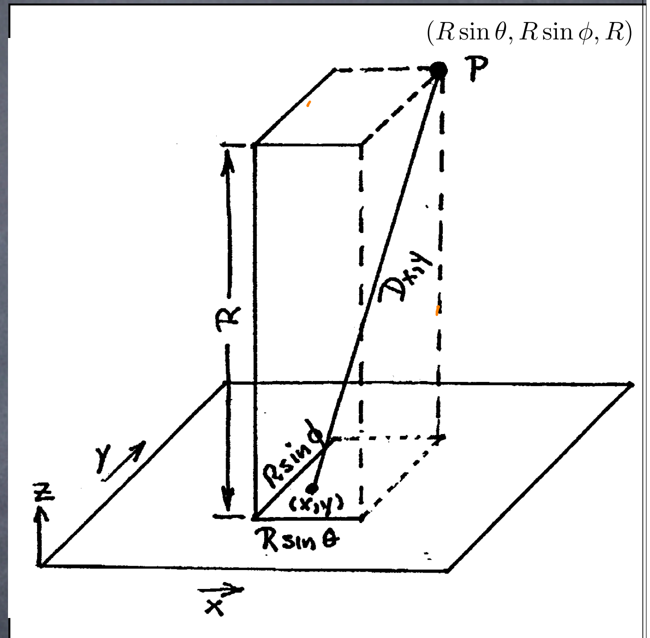
Large A_T w.r.t. λ^2 more concentrated beam

Large A_R, larger collection area

Inverse square law

Derivation of Friis Equation

- Consider a point P in the z=R plane in Cartesian 3D space
- Consider the electric field at P resulting from a differential element dx dy centered at (x,y) in the z=0 plane, having field



$$dF_{x,y}(P) = \frac{C}{D_{x,y}} \exp\left\{-i\frac{2\pi}{\lambda}D_{x,y}\right\}E(x,y)\,dx\,dy,$$

So we have

$$dF_{x,y}(P) = \frac{C}{D_{x,y}} \exp\left\{-i\frac{2\pi}{\lambda}D_{x,y}\right\}E(x,y)\,dx\,dy,$$

where

$$D_{x,y} = \sqrt{(R\sin\theta - x)^2 + (R\sin\phi - y)^2 + R^2}$$

$$= \text{distance from } (x,y,0) \text{ to } P.$$

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$$D_{x,y} = \sqrt{(R\sin\theta - x)^2 + (R\sin\phi - y)^2 + R^2}$$

Because R is arbitrarily large, while x and y are fixed and $x, y \ll R$, we can ignore the x^2 and y^2 terms in $D_{x,y}$.

$$D_{x,y} \cong \sqrt{R^2 + R^2\sin^2\theta - 2xR\sin\theta + R^2\sin^2\phi - 2yR\sin\phi}$$

Furthermore, if we use $\sqrt{1-\epsilon} \approx 1 - \epsilon/2$, we can write

$$D_{x,y} = R'\left(1 - \frac{x\sin\theta + y\sin\phi}{R(1+\sin^2\theta+\sin^2\phi)}\right)$$

$$R' = R\left(1 + \frac{1}{2}(\sin^2\theta + \sin^2\phi)\right)$$

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$$D_{x,y} = R'\left(1 - \frac{x\sin\theta + y\sin\phi}{R(1+\sin^2\theta+\sin^2\phi)}\right)$$

Because θ and ϕ are small, $|\sin\theta| \ll 1$ and $|\sin\phi| \ll 1$

substitute R'

$$D_{x,y} \cong R'\left(1 - \frac{x\sin\theta + y\sin\phi}{R\left(1+\frac{1}{2}(\sin^2\theta+\sin^2\phi)\right)}\right)$$

$$= R'\left(1 - \frac{x\sin\theta + y\sin\phi}{R'}\right)$$

$$= R' - x\sin\theta - y\sin\phi$$

$$D_{x,y} = R' - x\sin\theta - y\sin\phi$$

$$dF_{x,y}(P) = \frac{C}{D_{x,y}} \exp\left\{-i\frac{2\pi}{\lambda} D_{x,y}\right\} E(x,y)\,dx\,dy,$$

$$\Downarrow$$

$$dF_{x,y}(P) = \frac{C}{D_{x,y}} \exp\left\{-i\frac{2\pi}{\lambda}R'\right\} \exp\left\{\frac{i2\pi}{\lambda}\left[x\sin\theta + y\sin\phi\right]\right\} \cdot E(x,y)\,dx\,dy$$

$\Downarrow$ Replace $D_{x,y}$ by R

$$dF_{x,y}(P) = \frac{C}{R} \exp\left\{-i\frac{2\pi}{\lambda}R'\right\} \exp\left\{\frac{i2\pi}{\lambda}\left[x\sin\theta + y\sin\phi\right]\right\} \cdot E(x,y)\,dx\,dy$$

So we have

$$dF_{x,y}(P) = \frac{C}{R} \exp\left\{-i\frac{2\pi}{\lambda}R'\right\} \exp\left\{\frac{i2\pi}{\lambda}\left[x\sin\theta + y\sin\phi\right]\right\} \cdot E(x,y)\,dx\,dy$$

We get the total field $F(P)$ at P
by integrating over $z = 0$ plane w.r.t. x and y:

$$F(P) = \frac{C}{R} \exp\left\{\frac{-i2\pi}{\lambda}R'\right\} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,y) \exp\left\{\frac{i2\pi}{\lambda}(x\sin\theta + y\sin\phi)\right\} dx\,dy.$$

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$$F(P) = \frac{C}{R}\exp\left\{\frac{-i2\pi}{\lambda}R'\right\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E(x,y)\exp\left\{\frac{i2\pi}{\lambda}(x\sin\theta + y\sin\phi)\right\}dx\,dy.$$

Replace R' by R—interpreting as field at point P on a sphere of radius R centered at $(0,0,0)$

$$F(P) = \frac{C}{R}\exp\left\{\frac{-i2\pi}{\lambda}R\right\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E(x,y)\exp\left\{\frac{i2\pi}{\lambda}(x\sin\theta + y\sin\phi)\right\}dx\,dy.$$

This amounts to moving the points in slightly, but still gives the correct answer at P.

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Now consider a square aperture (wxw) uniformly illuminated by a field Eo in the z=0 plane (centered at origin)

$$F(P) = \frac{C}{R}\exp\left\{\frac{-i2\pi}{\lambda}R\right\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E(x,y)\exp\left\{\frac{i2\pi}{\lambda}(x\sin\theta + y\sin\phi)\right\}dx\,dy.$$

$$= \frac{C}{R}\exp\left\{\frac{-i2\pi}{\lambda}R\right\}\int_{-W/2}^{W/2}\int_{-W/2}^{W/2}E_0\exp\left\{i\left(\frac{2\pi}{\lambda}\right)x\sin\theta\right\}\exp\left\{i\left(\frac{2\pi}{\lambda}\right)y\sin\phi\right\}dx\,dy$$

$$= \frac{E_0CW^2}{R}\exp\left\{\frac{-i2\pi}{\lambda}R\right\}\left[\frac{\sin\left(\pi W\sin\theta/\lambda\right)}{\pi W\sin\theta/\lambda}\right]\left[\frac{\sin\left(\pi W\sin\phi/\lambda\right)}{\pi W\sin\phi/\lambda}\right]$$

$$\approx \frac{E_0CW^2}{R}\exp\left\{\frac{-i2\pi}{\lambda}R\right\}\left[\frac{\sin\left(\pi W\theta/\lambda\right)}{\pi W\theta/\lambda}\right]\left[\frac{\sin\left(\pi W\phi/\lambda\right)}{\pi W\phi/\lambda}\right] \quad \text{(for small } \theta \text{ and } \phi\text{)}$$

To find the power density at P, we compute

$$F(P)F(P)^* = \frac{|E_0|^2 C^2 W^4}{R^2}\left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]^2\left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right]^2$$

To find the total power in a region of the z=R plane at small angles about (0,0,R), we integrate this power density over that region w.r.t. P

Physical reasoning tells us that most of the power is at small angles if

$$W \gg \lambda$$

If we integrate over the whole sphere of radius R, we only get significant contributions at small angles

$$P_{\text{Total}} = \int\int_{S_R} F(P)F(P)^*\,dP$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|E_0|^2 C^2 W^4}{R^2}\left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]^2\left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right]^2 d(R\sin\theta)\,d(R\sin\phi)$$

note extension of limits!

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|E_0|^2 C^2 W^4}{R^2}\left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]^2\left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right]^2 R^2\,d\theta\,d\phi$$

(noting that $d(R\sin\theta)\,d(R\sin\phi) = R^2\cos\theta\cos\phi\,d\theta\,d\phi \approx R^2\,d\theta\,d\phi$ for small angles)

$$= \frac{\lambda^2|E_0|^2C^2W^4}{W^2}\left(\int_{-\infty}^{\infty}\left[\frac{\sin\pi u}{\pi u}\right]^2 du\right)\left(\int_{-\infty}^{\infty}\left[\frac{\sin\pi v}{\pi v}\right]^2 dv\right)$$

(Here $u = W\theta/\lambda$ and $v = W\phi/\lambda$)

Both integrals in parenthesis equal 1!

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So

$$P_{\text{Total}} = \lambda^2 |E_0|^2 C^2 W^2$$

But the total power leaving the transmit aperture is

$$P_{\text{Total}} = |E_0|^2 W^2$$

So by conservation of energy (power)

$$|E_0|^2 W^2 = \lambda^2 |E_0|^2 C^2 W^2$$

and thus

$$C = \frac{1}{\lambda}$$

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Substituting this C into out previous power density expression...

$$F(P)F(P)^* = \frac{|E_0|^2 W^4}{\lambda^2 R^2} \left[\frac{\sin(\pi W \theta / \lambda)}{\pi W \theta / \lambda} \right]^2 \left[\frac{\sin(\pi W \phi / \lambda)}{\pi W \phi / \lambda} \right]^2$$

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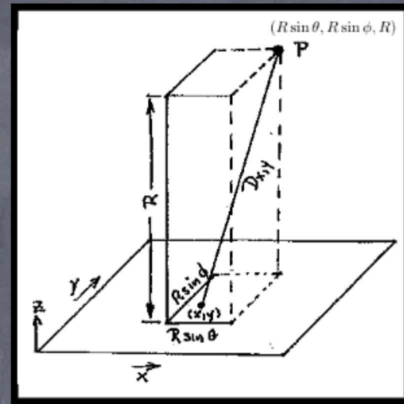
Recall that we showed that the power density at point P is



$$F(P)F(P)^* = \frac{|E_0|^2 C^2 W^4}{R^2} \left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]^2 \left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right]^2$$

and that the unknown constant C is

$$C = 1/\lambda$$

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$$F(P)F(P)^* = \frac{|E_0|^2 C^2 W^4}{R^2} \left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]^2 \left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right]^2$$

$$\Downarrow$$

$$C = 1/\lambda$$ Substituting

$$\Downarrow$$

$$F(P)F(P)^* = \frac{|E_0|^2 W^4}{\lambda^2 R^2} \left[\frac{\sin(\pi W\theta/\lambda)}{\pi W\theta/\lambda}\right]^2 \left[\frac{\sin(\pi W\phi/\lambda)}{\pi W\phi/\lambda}\right]^2$$

A ~~transmit~~ receive antenna with area A_R located at $(0,0,R)$ will receive power

$$A_R F(P)F^*(P) = \frac{|E_0|^2 A_R W^4}{\lambda^2 R^4} = \frac{A_R A_T^2 |E_0|^2}{\lambda^2 R^2}$$

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So the received power is

$$P_R = \frac{A_R A_T^2 |E_0|^2}{\lambda^2 R^2},$$

and the transmitted power is

$$P_T = |E_0|^2 A_T.$$

$$\Downarrow$$

$$\frac{P_R}{P_T} = \frac{\frac{A_R A_T^2 |E_0|^2}{\lambda^2 R^2}}{|E_0|^2 A_T} = \boxed{\frac{A_T A_R}{\lambda^2 R^2}}$$

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It appears we have derived this result for square transmit apertures and arbitrary receive apertures.

Reciprocity Theorem: For linear antenna systems, P_R/P_T remains the same when the roles of the transmit and receive antennas are reversed.

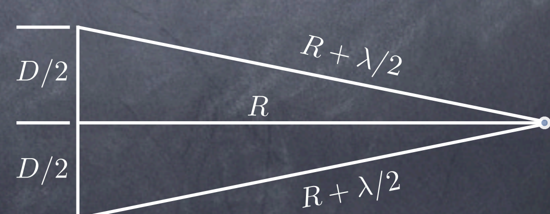
$\Longrightarrow$ P_R/P_T proportional to uniformly illuminated area A_T regardless of shape.

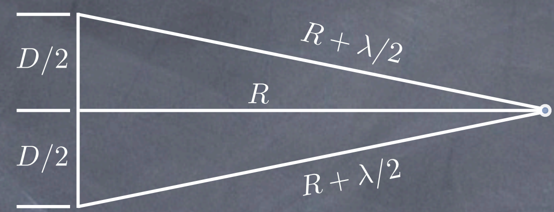
$$\therefore \quad \boxed{\frac{P_R}{P_T} = \frac{A_T A_R}{\lambda^2 R^2}}$$

- Friis equation derived using scalar diffraction—scalar wave equation describes propagation.
- Valid in other situations descibed by scalar wave equation:
- Acoustic radiation (in non-viscous media)
- Polarized light

- Friis equation holds when antennas are "far apart." How far?
- Uniformly illuminated aperture focuses objects at infinity.
- If we look at an antenna from close in, the edges appear farther away than the

A bad situation:





This bad situation occurs when

$$\left(R+\frac{\lambda}{2}\right)^2 = \left(\frac{D}{2}\right)^2 + R^2$$

$$\Longrightarrow \qquad R = \frac{D^2-\lambda^2}{4\lambda} \approx \frac{D^2}{4\lambda}, \qquad \text{because } D \gg \lambda$$

Thus we want

$$R \gg \frac{D^2}{4\lambda}$$

Maybe 10 times bigger—or more!

A common approach is to consider a point to be in the far-field when the center and edges of the aperture differ by no more than $\lambda/8$

If our calculations show

$$\frac{P_R}{P_T} = \frac{A_T A_R}{\lambda^2 R^2} > 1 \quad \Rightarrow \quad \text{Something is wrong!}$$

When such a situation occurs, a large fraction of the transmitted power is received by the receive aperture:

"Hertzian Cable"

This will not occur when R is such that the antennas are in each others far field.

Effective Area of an Antenna

- In deriving the Friis equation, we have assumed uniform aperture illumination.
- Practically, this is hard to achieve.
- When we do not have uniform illumination, the effective size of the aperture is smaller than the geometric area.
- How much smaller?

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Suppose $E(x, y)$ is the field over the aperture (i.e., $E(x, y) \neq E_0$.)

The power P_T flowing out over the aperture is

$$P_T = \frac{1}{Z_{FS}} \int\int_{A_{TG}} |E(x, y)|^2 \, dx \, dy,$$

The electric field at the receive aperture is

$$E_R = B \int\int_{A_{TG}} E(x, y) \, dx \, dy$$

↑ constant

The total power at the receive aperture is

$$P_R = \frac{|E_R|^2}{Z_{FS}} A_R = \frac{A_R B^2}{Z_{FS}} \left| \int\int_{A_{TG}} E(x, y) \, dx \, dy \right|^2 .$$

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Thus we have

$$\frac{P_R}{P_T} = B^2 A_R \frac{\left| \int\int_{A_{TG}} E(x, y) \, dx \, dy \right|^2}{\int\int_{A_{TG}} |E(x, y)|^2 \, dx \, dy}$$

Now define the *spatial average field* across A_{TG}

$$\overline{E} = \frac{1}{A_{TG}} \int\int_{A_{TG}} E(x, y) \, dx \, dy$$

and the *spatial mean-square field* across the aperture as

$$\overline{|E|^2} = \frac{1}{A_{TG}} \int\int_{A_{TG}} |E(x, y)|^2 \, dx \, dy$$

Hence

$$\frac{P_R}{P_T} = B^2 A_R A_{TG} \left[\frac{|\overline{E}|^2}{\overline{|E|^2}} \right]$$

To find B in

$$\frac{P_R}{P_T} = B^2 A_R A_{TG} \left[\frac{|\overline{E}|^2}{\overline{|E|^2}} \right]$$

we note that when $E(x,y) = E_0$ (uniform illum. $\Rightarrow A_{TG} = A_T$)

$$|\overline{E}|^2 = \overline{|E|^2} \qquad \Rightarrow \qquad B^2 = \frac{P_R}{P_T} \cdot \frac{1}{A_T A_R}$$

Thus A_T in the Friis equation is

$$A_T = A_{TG} \left[\frac{|\overline{E}|^2}{\overline{|E|^2}} \right] = \eta_T \cdot A_{TG}$$

and by reciprocity

$$A_R = A_{RG} \left[\frac{|\overline{E}|^2}{\overline{|E|^2}} \right] = \eta_R \cdot A_{RG}$$

Aperture Efficiency (η_T, η_R)

Effective Area

- Even antennas that don't have well defined apertures can be assigned effective areas for use in the Friis equation.
- Here we can use EM field theory to compute the field strength due to the transmit antenna at the receive aperture:

$$A_T = \frac{\lambda^2 R^2}{A_R} \cdot \frac{P_R}{P_T}$$

For example, for a half-wave dipole

$$A_T \approx 0.130\lambda^2$$

Antenna Gain over an Isotropic Radiator

- Isotropic Radiator: An antenna that radiates energy uniformly in all directions (transmit).
- On receive, it is equally sensitive to energy from all directions (by reciprocity).

If at a distance R from an isotropic radiator, we place a receive aperture A_R

$$\frac{P_R}{P_T} = \frac{A_R}{4\pi R^2}$$

surface area of sphere of radius R

$$\frac{P_R}{P_T} = \frac{A_R}{4\pi R^2} = \frac{A_i A_R}{\lambda^2 R^2} \quad \Rightarrow \quad \boxed{A_i = \frac{\lambda^2}{4\pi}}$$