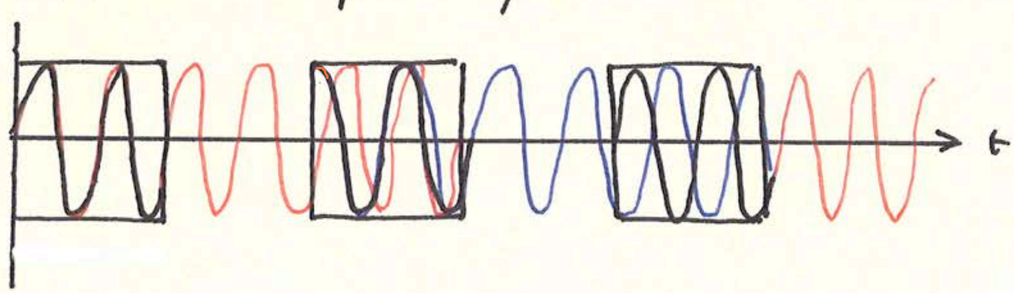
Session 25

Recall...

2. A Noncoherent pulse system



In case 2 – which is used in older and less expensive radar systems, the oscillator is effectively turned on with the transmission of each pulse. This is usually modeled by applying a complex phase factor $e^{i\theta_n}$ to each pulse at baseband, where $\theta_1, \ldots, \theta_N$ are modeled as i.i.d $U[0,2\pi)$.

Recall...

2. Radar Detection with a Noncoherent Pulse Train

25.2

- Assume that we once again detect a constant target, this time using an M pulse pulse train, but now assume there is a complete lack of coherence between pulses.
- Assume that once again, each pulse is processed with a complex baseband matched filter.

Under H_0: The complex matched filter output of each pulse is of the form

$$Z_m = X_m + iY_m \,, \quad X_m, Y_m \sim N[0, \sigma^2],$$
$$X_m \perp\!\!\!\perp Y_m \,,$$
$$(X_m, Y_m) \perp\!\!\!\perp (X_n, Y_n), \; n \neq m.$$

Recall that we were looking at the problem of detection using M noncoherent pulses:

25.3

As in the single noncoherent pulse case, each individual pulse return has pdf

$R_m = |Z_m|, \; m = 1, \dots, M.$

Under H_0:

$$f_{R_m,0}(r_m) = \frac{r_m}{\sigma^2} e^{-r_m^2/2\sigma^2} \cdot 1_{[0,\infty)}(r_m), \quad m = 1, 2, \dots, M.$$

(n.b., The R_m are conditionally i.i.d.)

Under H_1:

$$f_{R_m,1}(r_m) = \frac{r_m}{\sigma^2} \exp\left\{-\frac{(r_m^2 + A^2)}{2\sigma^2}\right\} I_0\left(\frac{r_m A}{\sigma^2}\right) \cdot 1_{[0,\infty)}(r_m),$$
$$m = 1, 2, \dots, M$$

25.4

Thus the log-likelihood ratio of $\underline{R} = (R_1, R_2, \ldots, R_M)$ is

$$\ell(\underline{R}) = \ln\left(\frac{f_{\underline{R},1}(\underline{R})}{f_{\underline{R},0}(\underline{R})}\right) = \ln\left(\frac{f_{R_1,1}(R_1)\cdots f_{R_M,1}(R_M)}{f_{R_1,0}(R_1)\cdots f_{R_M,0}(R_M)}\right)$$

$$= \sum_{m=1}^{M} \ln\left(\frac{f_{R_m,1}(R_m)}{f_{R_m,0}(R_m)}\right) = \sum_{m=1}^{M}\left[\ln\left(I_0\left(\frac{R_m A}{\sigma^2}\right)\right) - \frac{A^2}{2\sigma^2}\right]$$

$$= \sum_{m=1}^{M} \ln\left[I_0\left(\frac{R_m A}{\sigma^2}\right)\right] - \frac{MA^2}{2\sigma^2}$$

25.5

Recall..

For small x, $I_0(x) \cong 1 + x^2$, and for $y \simeq 1$, $\ln y \cong y - 1$

Thus for small signal to noise ratios, we have

$$\ln I_0\left(\frac{R_m A}{\sigma^2}\right) \cong \ln\left[1 + \left(\frac{R_m A}{\sigma^2}\right)^2\right]$$

$$\simeq \left(\frac{R_m A}{\sigma^2}\right)^2.$$

Thus it follows that

$$\ell(\underline{R}) \cong \sum_{m=1}^{M} R_m^2 \left(\frac{A}{\sigma^2}\right)^2 = \left(\frac{A}{\sigma^2}\right)^2 \sum_{m=1}^{M} R_m^2 - \frac{MA^2}{2\sigma^2}.$$

Thus we can implement the test in the form

$$\ell(\underline{R}) \cong \left(\frac{A}{\sigma^2}\right)^2 \sum_{m=1}^{M} R_m^2 - \frac{MA^2}{2} \underset{H_0}{\overset{H_1}{\gtrless}} \ell_0$$

$$\Rightarrow \sum_{m=1}^{M} R_m^2 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ell_0 + \frac{MA^2}{2\sigma^2}}{(A/\sigma^2)} = T_0$$

Hence for weak signals, we have the statistic

$$T(\underline{R}) \triangleq \sum_{m=1}^{M} R_m^2 ,$$

which is used in a threshold test.

Weak Signal Test:

$$\phi(\underline{R}) = \begin{cases} 1, & T(\underline{R}) > T_0 \\ 0, & T(\underline{R}) \le T_0 \end{cases}$$

where

$$T(\underline{R}) = \sum_{m=1}^{M} R_m^2 .$$

(Square-Law Detector)

25.8

(*) Alternatively, for $x >> 1$, we have
(strong signal case)

$$I_0(x) \stackrel{\sim}{=} \frac{e^x}{\sqrt{2\pi x}}, \qquad (x>>1)$$

and thus

$$\ln\left[I_0\left(\frac{R_m A}{\sigma^2}\right)\right] \simeq \ln\left[\exp\left[\frac{R_m A}{\sigma^2}\right]\right] - \frac{1}{2}\ln\left(\frac{2\pi R_m A}{\sigma^2}\right)$$

$$\simeq \frac{R_m A}{\sigma^2}$$

$$\Rightarrow \quad l(\underline{R}) \stackrel{\sim}{=} \sum_{m=1}^{M} \frac{R_m A}{\sigma^2} - \frac{MA}{2\sigma^2} = \frac{A}{\sigma^2}\sum_{m=1}^{M} R_m - \frac{MA}{2\sigma^2}$$

$$\Rightarrow \quad \sum_{m=1}^{M} R_m \mathop{\gtrless}_{H_0}^{H_1} \frac{l_0 + \frac{MA}{2\sigma^2}}{A/\sigma^2} =: u_0$$

25.9

Thus in the strong signal case, we have a threshold test of the form

$$\phi(\underline{R}) = \begin{cases} 1, & u(\underline{R}) > u_0 \\ 0, & u(\underline{R}) \leq u_0 \end{cases}$$

where

$$u(\underline{R}) = \sum_{m=1}^{M} R_m$$

(Linear Detector)

So we have two integration rules:

Linear Detector: (Strong Signal case)

$$u(\underline{R}) = \sum_{m=1}^{M} R_m \underset{H_0}{\overset{H_1}{\gtrless}} u_0$$

Square-Law Detector: (Weak Signal Case)

$$T(\underline{R}) = \sum_{m=1}^{M} R_m^2 \underset{H_0}{\overset{H_1}{\gtrless}} T_0$$

In order to characterize the performance of these detectors, we need to find the distributions of $u(\underline{R})$ and $T(\underline{R})$ under both H_0 and H_1.

For the Weak Signal Case (Square-Law Detector)

The distribution of the sum $T(\underline{R})$ can be found as an M-fold convolution of these distributions (or use characteristic fzns.)

Under H_0:

$$f_{T,0}(t) = \frac{t^{(M-1)/2}}{2^M \sigma^{2M} (M-1)!} \exp\left(\frac{-t}{2\sigma^2}\right) 1_{[0,\infty)}(t)$$

Under H_1:

$$f_{T,1}(t) = \frac{1}{2\sigma^2}\left(\frac{t}{MA^2}\right)^{M-1} \cdot \exp\left(-\frac{(t+MA^2)}{2\sigma^2}\right) \cdot I_{M-1}\left(\frac{A\sqrt{t}}{\sigma^2}\right) \cdot 1_{[0,\infty)}(t)$$

25.12

Computing the likelihood ratio, we have

$$L(r) = \frac{f_{T,1}(t)}{f_{T,0}(t)} = \frac{2^M \sigma^2 (M-1)!}{(MA^2)^{M-1}} \exp\left\{-\frac{MA^2}{2\sigma^2}\right\} t^{(M-1)/2} I_{M-1}\left(\frac{A\sqrt{t}}{\sigma^2}\right)$$

$$= \underbrace{K(M,A)}_{\text{contant}} \cdot \underbrace{t^{(M-1)/2}}_{\text{monotone increasing in } t} \cdot \underbrace{I_{M-1}\left(\frac{A\sqrt{t}}{\sigma^2}\right)}_{\text{monotone increasing in } t} \underset{H_0}{\overset{H_1}{\gtrless}} L_0$$

$$\Rightarrow \quad T(R) \underset{H_0}{\overset{H_1}{\gtrless}} T_0$$

So this test is a threshold test on $T(R)$.

25.13

Given this fact and the distributions on $T(\underline{R})$ under H_0 and H_1, we can write

$$\alpha(M,T_0) = P_{FA}(M,T_0) = \int_{T_0}^{\infty} f_{T,0}(t)\,dt$$

$$\beta(M,T_0) = P_D(M,T_0) = \int_{T_0}^{\infty} f_{T,1}(t)\,dt$$

These can be evaluated numerically, and then for fixed M we can plot the ROC as a parametric plot

$$\{(\alpha(M,T_0), \beta(M,T_0)),\ T_0 \in [0,\infty)\}.$$

Performance of Square-Law Detector

The following numerically integrated curves are from

J.V. DiFranco and W.L. Rubin, *Radar Detection*, Prentice-Hall, 1968.

(Reprinted by Scitech, 2004)

In plots, R_p = Single pulse SNR $= \frac{2E_p}{N_0}$ E_p = energy per pulse.

n.b $n' =$ "false alarm number" $\triangleq \frac{0.693}{P_{FA}}$

$\Rightarrow P_{FA} = \frac{0.693}{n'}$ (Also $M = N$ in plots.)

Square-Law Detector Noncoherent Pulse Train Performance

R_p = Single pulse SNR $= \frac{2E_p}{N_0}$

M = 1

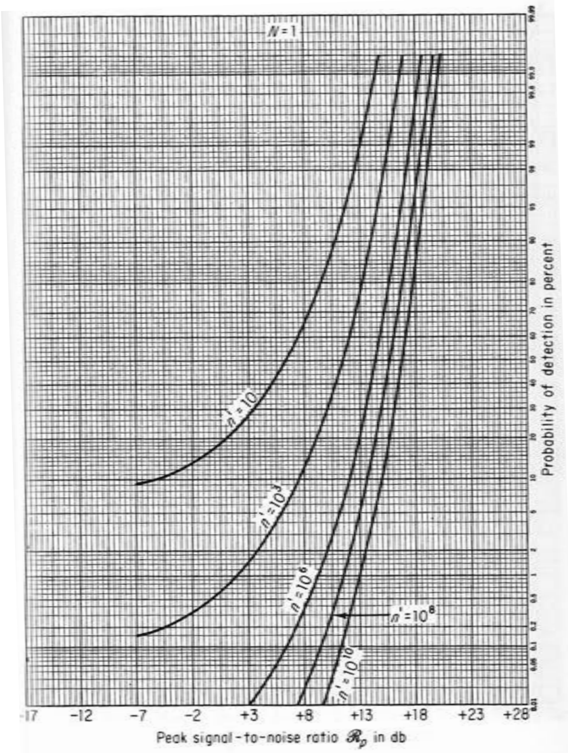


Fig. 10.4-1. Probability of detecting a nonfluctuating target (square-law detector), $N = 1$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

M = 2

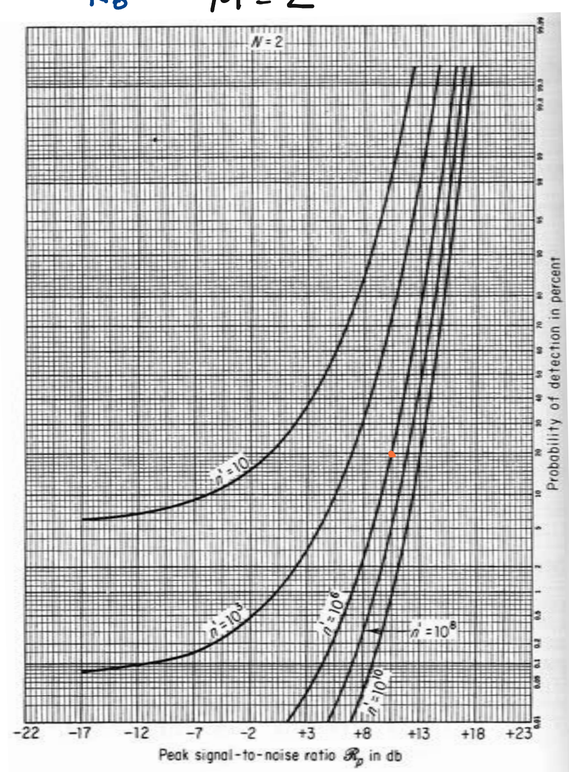


Fig. 10.4-2. Probability of detecting a nonfluctuating target (square-law detector), $N = 2$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

Square-Law Detector Noncoherent Pulse Train Performance

M = 3

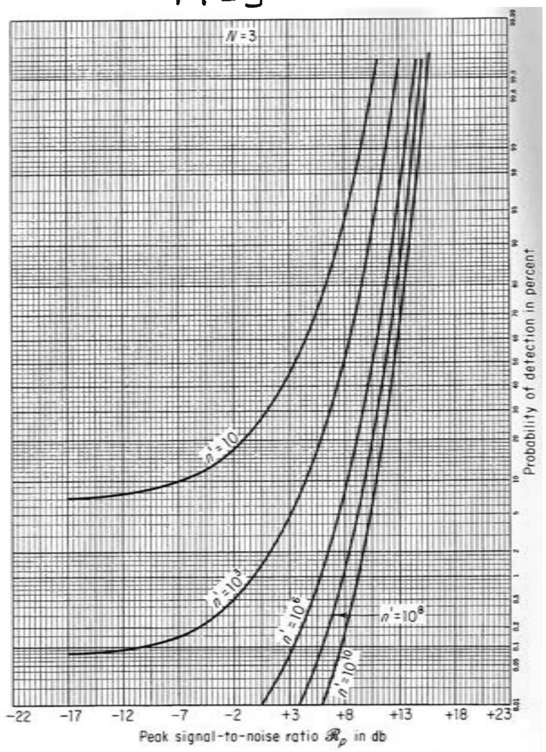


Fig. 10.4-3. Probability of detecting a nonfluctuating target (square-law detector), $N = 3$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

M = 6

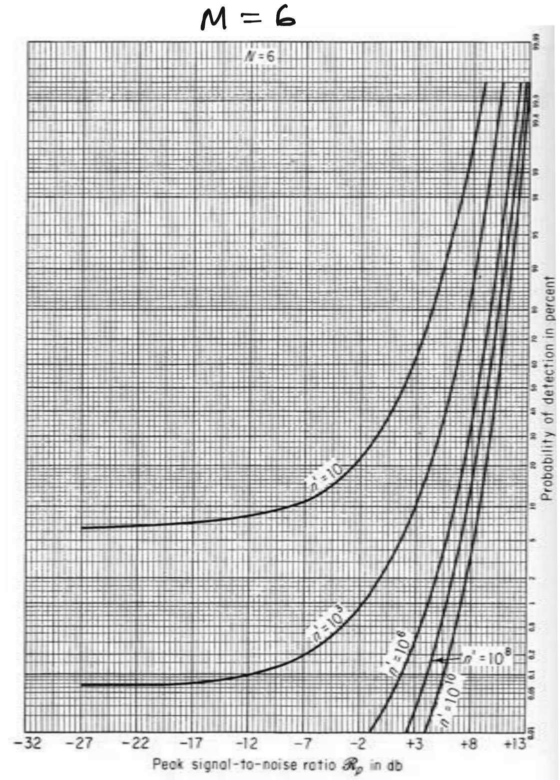


Fig. 10.4-4. Probability of detecting a nonfluctuating target (square-law detector), $N = 6$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

Square-Law Detector Noncoherent Pulse Train Performance

M = 10

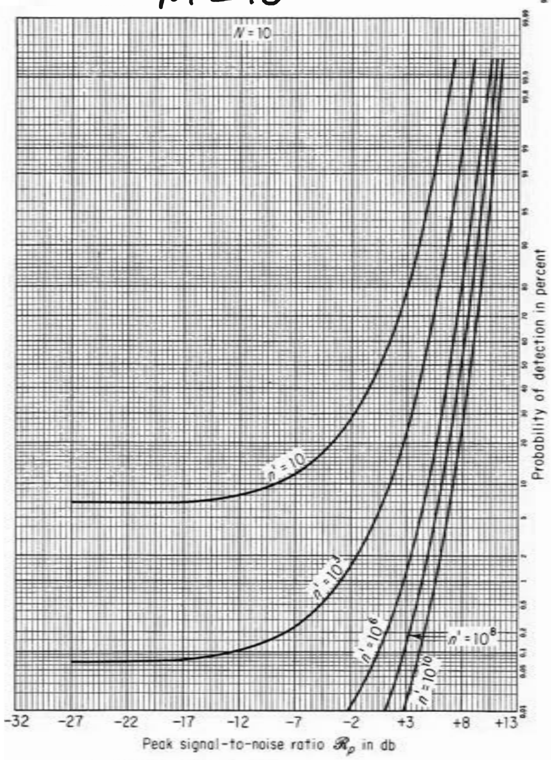


Fig. 10.4-5. Probability of detecting a nonfluctuating target (square-law detector), $N = 10$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

M = 30

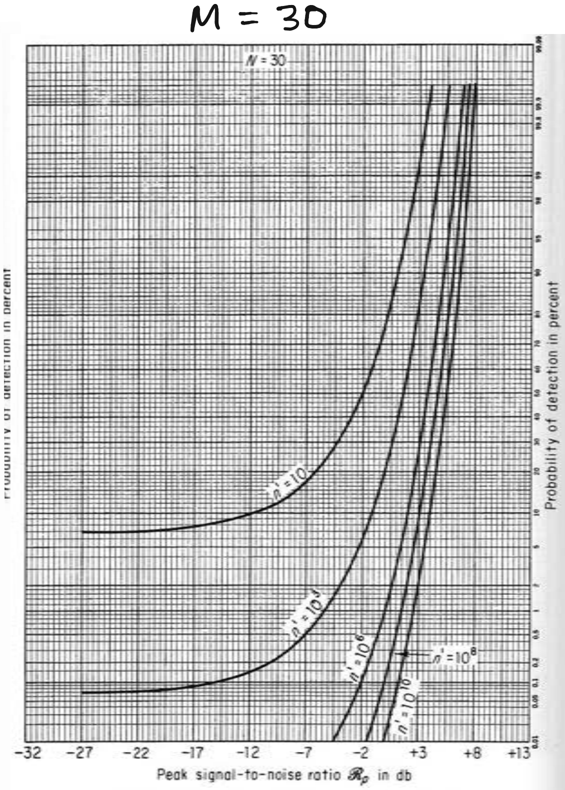


Fig. 10.4-6. Probability of detecting a nonfluctuating target (square-law detector), $N = 30$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

Square-Law Detector Noncoherent Pulse Train Performance

M = 100

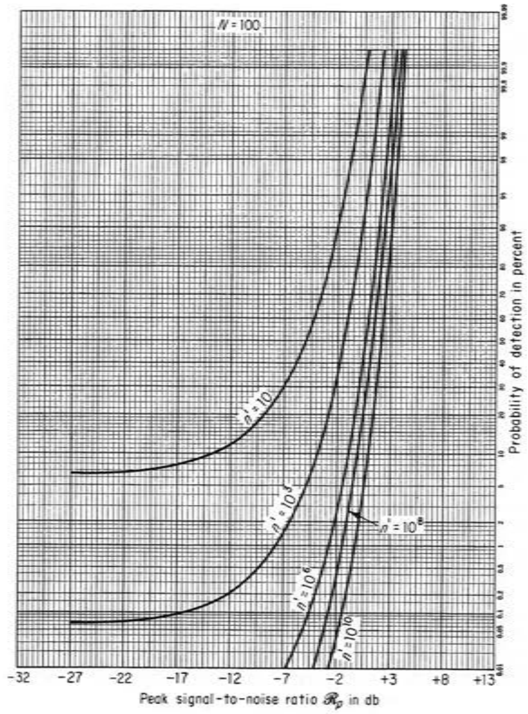
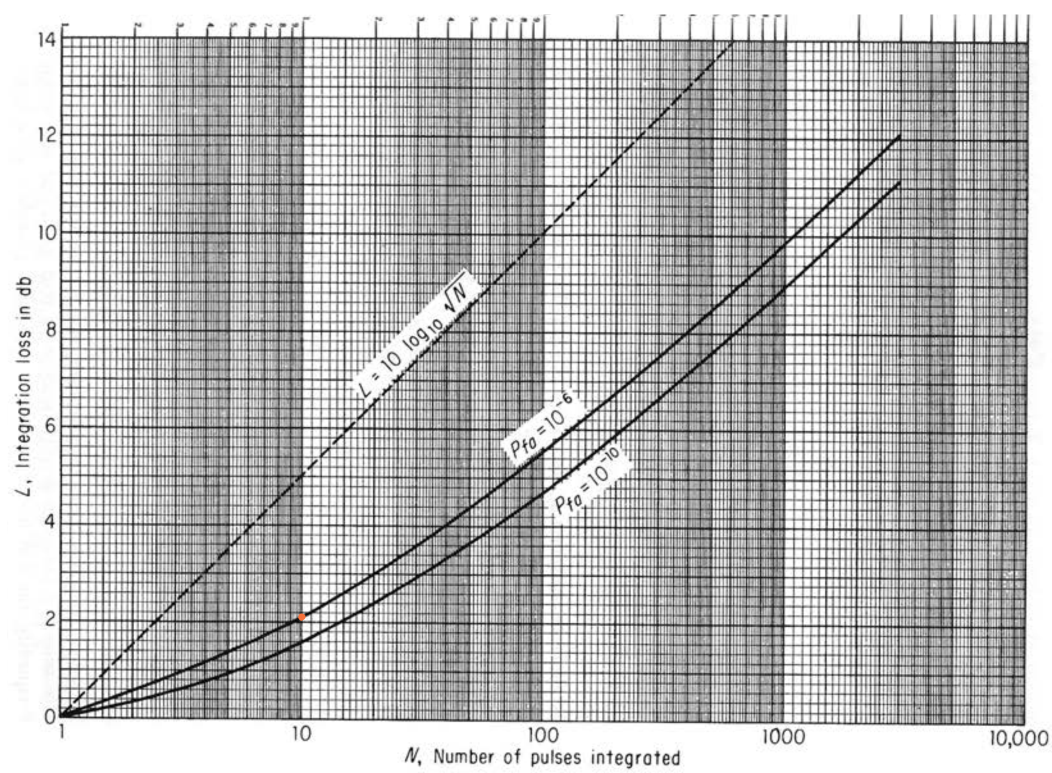


Fig. 10.4-7. Probability of detecting a nonfluctuating target (square-law detector), $N = 100$. [N = number of pulses incoherently integrated; $P_{fa} = 0.693/n'$.]

Non-Coherent Integration Loss Compared to Coherent Pulse Integration



Stretch Processing for Low-Complexity Radar Signal Processing

Wideband waveforms using LFM, phase coding or frequency coding are used in radar when high range resolution is needed.

These waveforms are often processed using a digital matched filter.

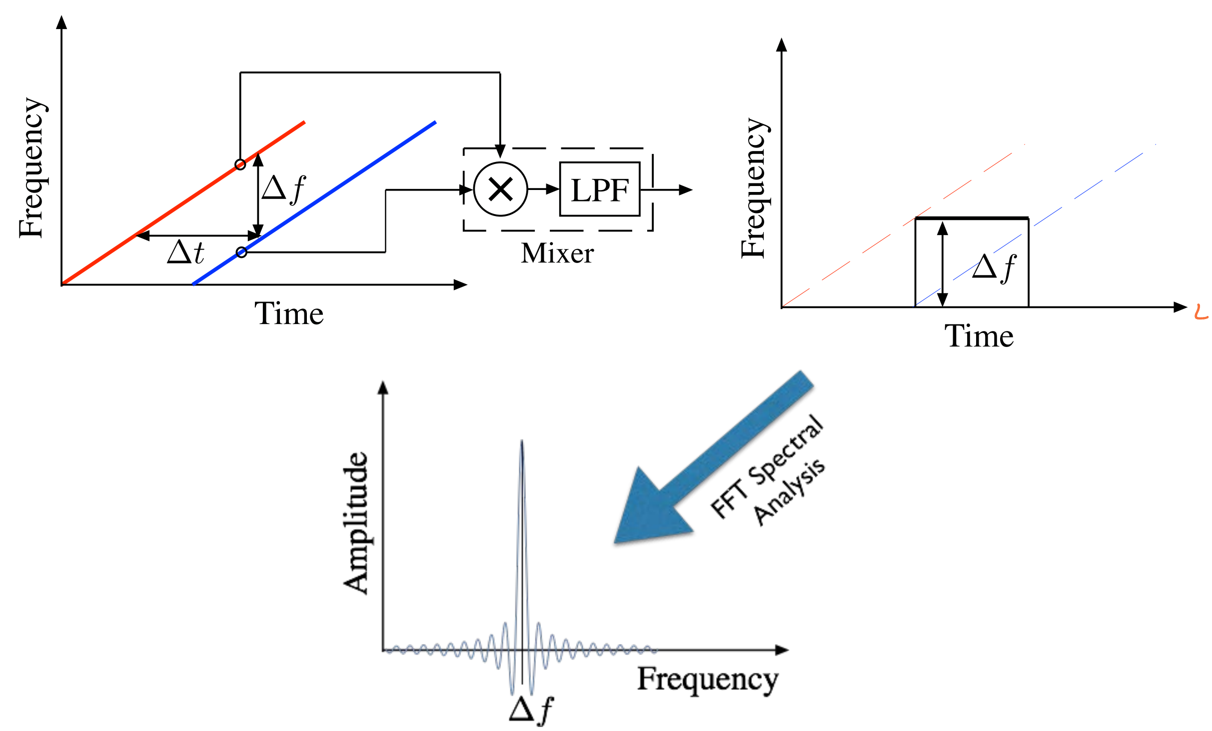
This requires sampling of the received signal at very high sampling rates (twice the waveform bandwidth.)

Stretch-processing makes use of the distinct structure of LFM waveforms to reduce the required sampling rate.

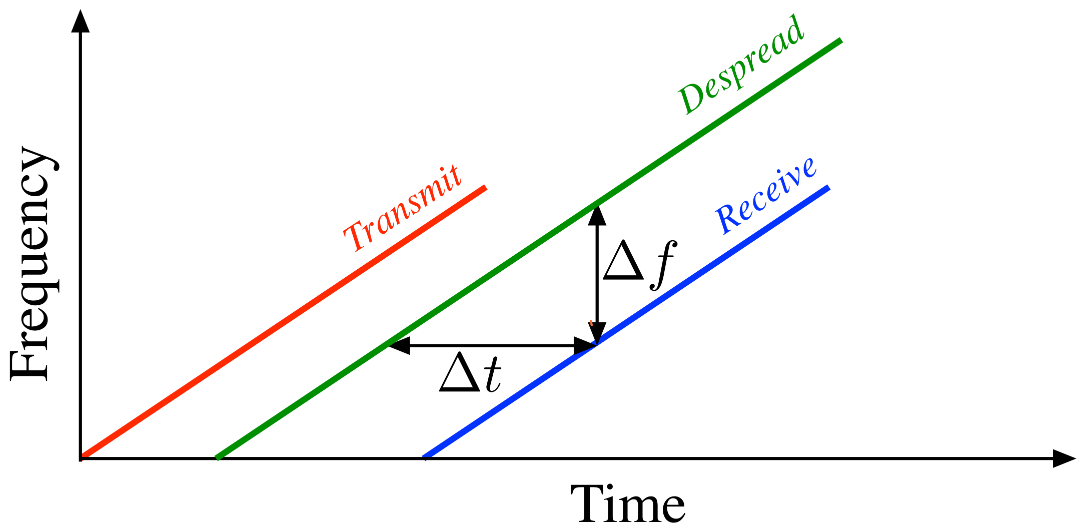
Stretch Processing allows for high resolution LFM measurements over a restricted range interval with much lower sampling rates than sampling the waveform bandwidth at the Nyquist rate as required by a digital matched filter.

Properly implemented, the range resolution is equivalent to that of the matched filter and the SNR is equivalent to the matched filter response of the LFM waveform.

The Basic Idea of Stretch Processing



Stretch Processing can be Modified for Additional Flexibility



Automotive Radar Example (Rohling)

An example from real world radar waveform:

A chirp waveform of 2.5milliseconds and 150MHz bandwidth is considered here. Chirp slope $k = \frac{150\text{Mhz}}{2.5\text{ms}}$.(data taken from 'waveform design principles for Automotive Radar System' by Hermann Rohling)

Match Filter: Sampling Frequency = 2x150Mhz = 300Mhz
#samples/chirp = 2.5ms × 300Mhz = 750,000 samples

Stretch Processing: Consider maximum tracking distance 400 meters, (At highway speed of 65 mph, recommended 3 second seperation means range of 100 meters.) $\Delta f = k\Delta t = \frac{150\text{Mhz}}{2.5\text{ms}} \times \frac{2 \times 400}{3 \times 10^8} = 80Khz$

$\Rightarrow$ Sampling Frequency = 2x80Khz = 160Khz

Stretch Processing

Transmit : $s(t) = e^{i\pi\alpha t^2} \cdot 1_{[0,T]}(t)$ (Complex Baseband)

Receive :
$$r(t) = \alpha_1 s(t-\tau_1)$$
$$= \alpha_1 e^{+i\pi\alpha(t-\tau_1)^2} \cdot 1_{[0,T]}(t-\tau_1)$$
$$= \alpha_1 e^{i\pi\alpha\tau_1^2} \cdot e^{i\pi\alpha t^2} \cdot e^{-i2\pi\alpha\tau_1 t} \cdot 1_{[\tau_1, T+\tau_1]}(t)$$

Multiplying the received signal by $e^{-i\pi\alpha t^2}$, we have

$$\tilde{r}(t) = r(t)e^{-i\pi\alpha t^2}$$
$$= \underbrace{\left(\alpha_1 e^{i\pi\alpha t_1^2}\right)}_{\text{Complex Constant}} \underbrace{e^{-i2\pi(\alpha\tau_1)t}}_{\text{Complex Sinusoid of Frequency } -\alpha\tau_1} \cdot 1_{[\tau_1, T+\tau_1]}(t)$$

n.b. Delay too and from target is proportional to observed frequency.

If we compute the Fourier Transform of $\tilde{r}(t)$, we get

$$\tilde{R}(f) = \int_{-\infty}^{\infty} \tilde{r}(t) e^{-i2\pi f t}\, dt$$
$$= \alpha_1 e^{i\pi\alpha\tau_1^2} \cdot \int_{\tau_1}^{T+\tau_1} e^{-i2\pi(\alpha\tau_1)t} e^{-i2\pi f t}\, dt$$

$(\text{Let } p = t-\tau_1 \Rightarrow t = p+\tau_1 \Rightarrow dp = dt)$

$$= \alpha_1 e^{i\pi\alpha\tau_1^2} \int_0^T e^{-i2\pi(\alpha\tau_1+f)(p+\tau_1)}\, dp$$
$$= \alpha_1 e^{i\pi\alpha\tau_1^2} e^{-i2\pi(\alpha\tau_1+f)\tau_1} \int_0^T e^{-i2\pi(\alpha\tau_1+f)p}\, dp$$
$$= \alpha_1 e^{i\pi\alpha\tau_1^2} e^{-i2\pi(\alpha\tau_1+f)} \cdot T e^{-i\pi(\alpha\tau_1+f)} \cdot \left[\frac{\sin \pi T(\alpha\tau_2+f)}{\pi T(\alpha\tau_1+f)}\right]$$

In order to separate two targets separated by $\Delta t = \tau_2 - \tau_1$, we must be able to resolve two sinusoids separated in frequency by $\alpha\Delta t$ using an observation window of length T.

This means that

$$\alpha \Delta t \geq \frac{1}{T} \quad \text{(Rayleigh Resolution Criterion)}$$

$$\Rightarrow \quad \Delta t \geq \frac{1}{\alpha T}$$

But the bandwidth of the transmitted signal is

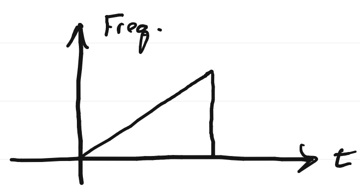
$$B \cong \alpha T$$

So stretch processing has resolution

$$\boxed{\Delta t \cong \frac{1}{B}}$$

This is also the resolution provided by the matched filter.

We've looked at how stretch processing is used for a single LFM chirp:



It can also be extended to CW LFM waveforms: