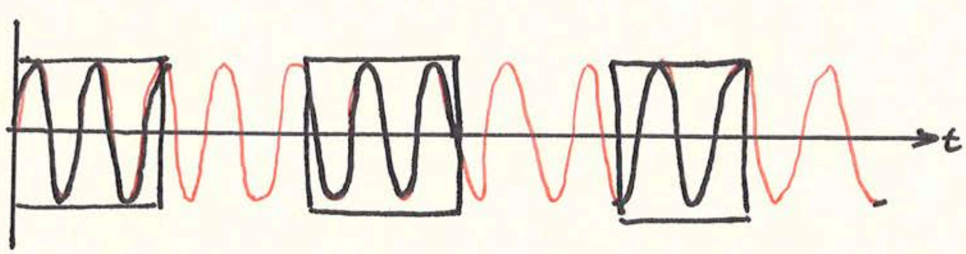
Session 24

Recall...

1. A coherent-pulse system



Case 1 in which pulse-to-pulse coherence is maintained is essential for pulse-Doppler radar operation. It allows us to view the phase shift from pulse-to-pulse caused by target motion.

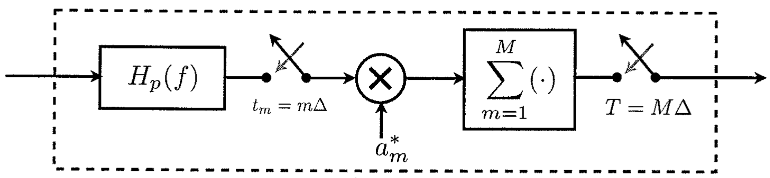
Recall...

So the matched filter can be written as

$$
\begin{aligned}
\tilde{H}_M(f) &= \frac{S^*(f)}{S_{nn}(f)} e^{-i2\pi f M\Delta} \\
&= \sum_{m=1}^{M} a_m^* \frac{P^*(f)}{S_{nn}(f)} e^{-i2\pi f \Delta} e^{-i2\pi (M-m) f\Delta} \qquad a_m = 1, \ \forall m \\
&= \sum_{m=1}^{M} a_m^* H_p(f) e^{-i2\pi (M-m) f \Delta},
\end{aligned}
$$

where $H_p(f)$ is the single pulse matched filter sampled at time $t = \Delta$ and given by

$$H_p(f) = \frac{P^*(f)}{S_{nn}(f)} e^{-i2\pi f \Delta}.$$



$\tilde{H}_M(f)$

Recall...

So we see that the matched-filter for the pulse train is obtained by summing the responses for the matched-filter of each pulse:

For a completely known signal

$$H_0 : \quad x(T) \sim N\left[0, \frac{2E_s M}{N_0}\right]$$

Because you are adding M independent zero-mean Gaussian RVs with mean 0 and variance $\frac{2E_s}{N_0}$.

$$H_1 : \quad x(T) \sim N\left[\frac{2E_s M}{N_0}, \frac{2E_s M}{N_0}\right]$$

Because you are adding M independent Gaussian RVs with mean $\frac{2E_s}{N_0}$ and variance $\frac{2E_s}{N_0}$.

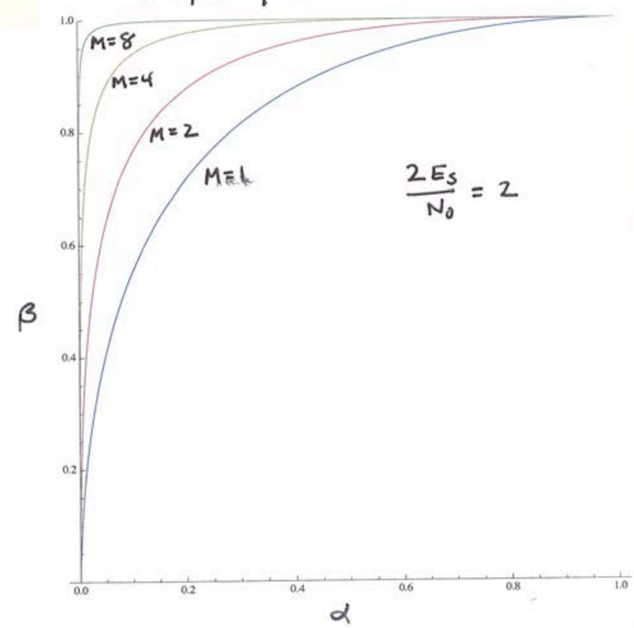
n.b. The standard deviation of the noise grows like $\sqrt{M}$ while the mean grows like M.
SNR increases by $\left(\frac{M}{\sqrt{M}}\right)^2 = \frac{M^2}{M} = \boxed{M}$.

Recall:

Thus we know that in this case, the ROC is given by

$$\beta(\alpha) = 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \sqrt{\frac{2ME_s}{N_0}}\right)$$

ROC For Detection of M Completely Known Pulses



Coherent Pulse-Train Detection with Identical Pulses

Unknown identical Phase for Each Pulse

The magnitude of the complex matched-filter output is taken as the decision statistic.

Under H_0: This corresponds to the sum of M independent circular complex Gaussian RVs:

$$x(T) = z_1 + z_2 + \cdots + z_M$$

Under H_1:

$$x(T) = e^{i\theta}\frac{2E_s}{N_0} + z_1 + e^{i\theta}\frac{2E_s}{N_0} + z_2 + \cdots + e^{i\theta}\frac{2E_s}{N_0} + z_M$$

$$= e^{i\theta}\cdot\frac{2ME_s}{N_0} + \underbrace{(z_1 + z_2 + \cdots + z_M)}$$

Zero-mean complex Gaussian with I and Q components having variance

$$\sigma_M^2 = M\sigma^2 = \frac{2ME_s}{N_0}.$$

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This looks just like the corresponding single pulse problem with a new mean and variance.

Thus it follows that we can write

$$\alpha(\gamma_0) = \int_{\gamma_0}^{\infty} z e^{-z^2/2}\, dz = e^{-\gamma_0^2/2}$$

$$\beta(\gamma_0) = \int_{\gamma_0}^{\infty} z e^{-\left(z^2 + \frac{2ME_s}{N_0}\right)} \cdot I_0\left(z\sqrt{\frac{2ME_s}{N_0}}\right) dz$$

$$= Q\left(\sqrt{\frac{2ME_s}{N_0}}, \gamma_0\right)$$

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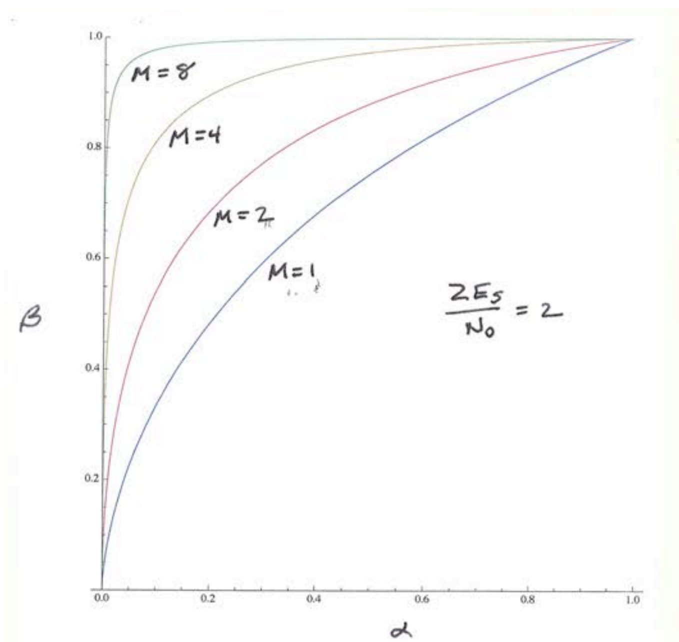
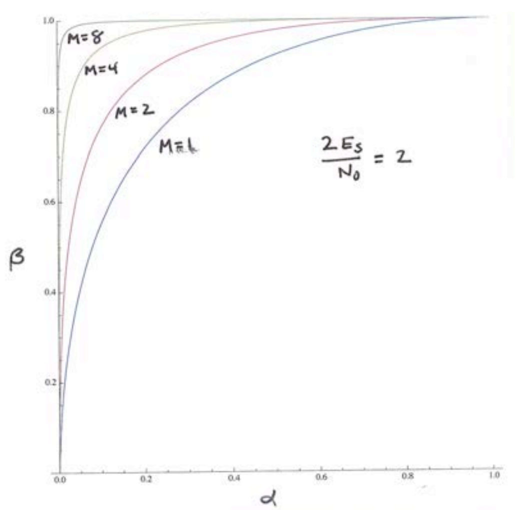
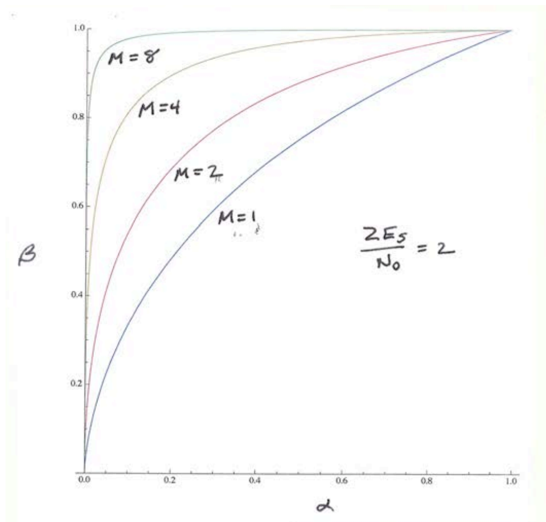
$$\beta(\alpha) = Q\left(\sqrt{\frac{2ME_s}{N_0}}, \sqrt{-2\ln\alpha}\right)$$



Coherent Pulse Train



Noncoherent Pulse Train

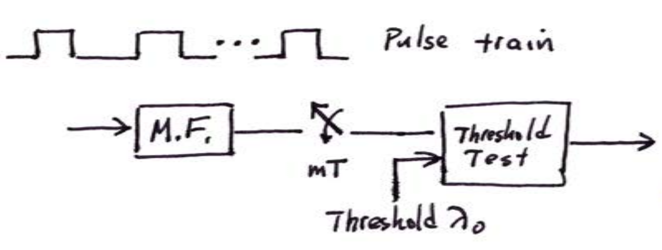


Binary Integration (See pp. 57–63 of Levanon, *Radar Principles*)

Another form of multi-pulse integration that – while weak – is often used is Binary Integration.

Idea: Send M pulses and detect each pulse separately and independently.

- Whenever a reflected pulse (target) is detected, Assign a 1.
- When no target is detected, assign a 0.



We get a stream of binary numbers.

In binary integration, we declare a target present if there are h or more "hits" (i.e., "1's") in the M pulse detections.

Assuming that the noise is stationary, we have that for each pulse

$$\alpha = P_{FA}^{(1)} = \alpha(\lambda_0)$$

$$\beta = P_D^{(1)} = \beta(\lambda_0)$$

Thus it follows that the overall M-pulse false alarm and detection probabilities are

$$P_{FA}(M) = \sum_{m=h}^{M} \binom{M}{m} \alpha^m (1-\alpha)^{M-m}$$

$$P_D(M) = \sum_{m=h}^{M} \binom{M}{m} \beta^m (1-\beta)^{M-m}$$

So in binary integration, we have two thresholds to select:

1. The single pulse detection threshold λ_0.
2. The "hit" number threshold h.

There are often several (λ_0, h) pairs that will yield a particular false alarm $P_F(M)$.

$\boxed{\text{We will want to select the combination } (\lambda_0, k) \text{ that gives the largest } P_D(M).}$

A special case: $h=1 \Rightarrow \underbrace{P_{cum}}$, Analysis

Cumulative Prob. of Detection

If we only need one success, we have

$$P_{FA}(M) = 1-(1-\alpha)^M = 1-(1-\alpha(\lambda_0))^M$$

$$P_D(M) = 1-(1-\beta)^M = 1-(1-\beta(\lambda_0))^M$$

If $\alpha \ll 1$ (and $\alpha \ll \beta$), then

$$P_{FA}(M) \simeq M\alpha = M\alpha(\lambda_0)$$

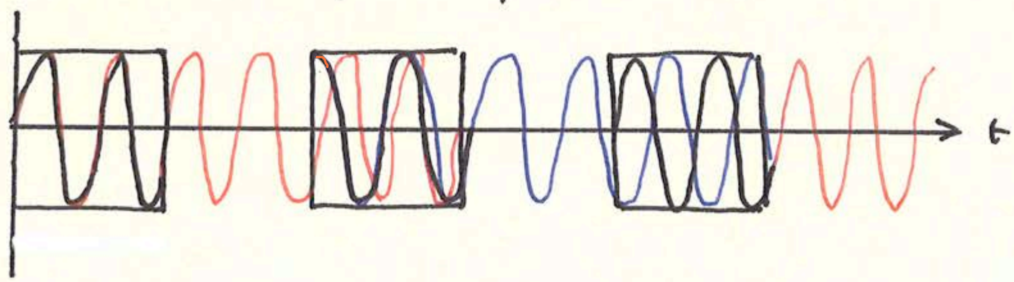
$$P_D(M) = 1-(1-\beta)^M = 1-(1-\beta(\lambda_0))^M$$

If $\alpha \ll 1$ and $0.25 < \beta < 0.5$, you can get very reasonable results.

Table 3.1 Required Detection Probability for Cumulative Detection Probability of 0.99

M	1	2	3	4	5	6	10	20	100
$P_D \vert_{P_{CD}=0.99}$	0.99	0.90	0.78	0.68	0.60	0.54	0.37	0.20	0.045

2. A Noncoherent pulse system



In case 2 – which is used in older and less expensive radar systems, the oscillator is effectively turned on with the transmission of each pulse. This is usually modeled by applying a complex phase factor $e^{i\Theta_n}$ to each pulse at baseband, where $\Theta_1, \ldots, \Theta_N$ are modeled as i.i.d $U[0,2\pi)$.

2. Radar Detection with a Noncoherent Pulse Train

Assume that we once again detect a constant target with an M pulse pulse-train, but now assume there is a complete lack of coherence between pulses.

Assume that once again each pulse is processed using a matched filter. Under hypothesis H_0, we have

Under H_0:

The complex matched filter output of each pulse is of the form

$$Z_m = X_m + iY_m, \quad X_m, Y_m \sim N[0, \sigma^2]$$

$$X_m \perp\!\!\!\perp Y_m$$

$$(X_m, Y_m) \perp\!\!\!\perp (X_n, Y_n), \quad m \neq n$$

$$m = 1, 2, \ldots, M.$$

Under H_1:

$$Z_m = \underbrace{\frac{2E}{N_0}}_{A} e^{i\theta_m} + \underbrace{X_m + iY_m}_{\substack{\text{complex} \\ \text{Gaussians} \\ \text{as above}}}$$

$$\theta_1, \theta_2, \dots, \theta_m \quad \text{i.i.d.} \quad U[0, 2\pi)$$

As before, the phase does not contain useful information for the decision process.

Thus we base the decision on the conditionally i.i.d. RVs $R_1, R_2, \dots, R_M$.

where $R_m = |Z_m|$, $m = 0, \dots, M$

As in the single pulse case

Under H_0:

$$f_{R_m,0}(r_m) = \frac{r_m}{\sigma^2} e^{-r_m^2/2\sigma^2} \cdot 1_{[0,\infty)}(r_m), \quad m = 1, 2, \dots, M.$$

Under H_1:

$$f_{R_m,1}(r_m) = \frac{r_m}{\sigma^2} \exp\left\{ \frac{-(r_m^2 + A^2)}{2\sigma^2} \right\} \cdot I_0\left(\frac{r_m A}{\sigma^2}\right) \cdot 1_{[0,\infty)}(r_m)$$

$$m = 1, 2, \dots, M$$

Now for small x, $I_0(x) \cong 1 + x^2$
and for $y \simeq 1$, $\ln(y) \cong y - 1$
Thus for small signal-to-noise ratios

$$\log I_0\left(\frac{R_m A}{\sigma^2}\right) \cong \log\left(1 + \left[\frac{R_m A}{\sigma^2}\right]^2\right)$$

$$\cong R_m^2 \left(\frac{A}{\sigma^2}\right)^2$$

Thus it follows that

$$\ell(\underline{R}) \cong \sum_{m=1}^{M} R_m^2 \left(\frac{A}{\sigma^2}\right)^2 - \frac{MA^2}{2\sigma^2} = \left(\frac{A}{\sigma^2}\right)^2 \sum_{m=1}^{M} R_m^2 - \frac{MA^2}{2\sigma^2}$$

Thus the log-likelihood ratio for $\underline{R} = (R_1, \ldots, R_M)$ is

$$\ell(\underline{R}) = \ln\left(\frac{f_{\underline{R},1}(\underline{R})}{f_{\underline{R},0}(\underline{R})}\right) = \ln\left(\frac{f_{R_1,1}(R_1) \cdot f_{R_2,1}(R_2) \cdots f_{R_M,1}(R_M)}{f_{R_1,0}(R_1) \cdot f_{R_2,0}(R_2) \cdots f_{R_M,0}(R_M)}\right)$$

$$= \sum_{m=1}^{M} \log\left(\frac{f_{R_m,1}(R_m)}{f_{R_m,0}(R_m)}\right) = \sum_{m=1}^{M}\left[\log I_0\left(\frac{R_m A}{\sigma^2}\right) - \frac{A^2}{2\sigma^2}\right]$$

$$= \sum_{m=1}^{M} \log I_0\left(\frac{R_m A}{\sigma^2}\right) - \frac{MA^2}{2\sigma^2}$$

Thus we can implement the test in the form

$$l(\underline{R}) \cong \left(\frac{A}{\sigma^2}\right)^2 \sum_{m=1}^{M} R_m^2 - \frac{MA^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} l_0$$

$$\Rightarrow \sum_{m=1}^{M} R_m^2 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{l_0 + \frac{MA^2}{2\sigma^2}}{(A/\sigma^2)^2} = \gamma_0$$

Hence for weak signals, the statistic

$$T(\underline{R}) = \sum_{m=1}^{M} R_m^2$$

is used in the threshold test

$$\boxed{\begin{aligned} \phi(\underline{R}) &= \begin{cases} 1, & T(\underline{R}) > \gamma_0 \\ 0, & T(\underline{R}) \le \gamma_0 \end{cases} \\ T(\underline{R}) &= \sum_{m=1}^{M} R_m^2 \end{aligned}}$$

Weak Signal Test

Alternatively, for $x >> 1$, we have

$$I_0(x) \cong \frac{e^x}{\sqrt{2\pi x}}, \qquad (x >> 1)$$

and thus

$$\ln\left[I_0\left(\frac{R_m A}{\sigma^2}\right)\right] \simeq \ln\left[\exp\left[\frac{R_m A}{\sigma^2}\right]\right] - \frac{1}{2}\ln\left(\frac{2\pi R_m A}{\sigma^2}\right)$$

$$\simeq \frac{R_m A}{\sigma^2}$$

$$\Rightarrow l(\underline{R}) \simeq \sum_{m=1}^{M} \frac{R_m A}{\sigma^2} - \frac{MA}{2\sigma^2} = \frac{A}{\sigma^2}\sum_{m=1}^{M} R_m - \frac{MA}{2\sigma^2}$$

$$\Rightarrow \sum_{m=1}^{M} R_m \underset{H_0}{\overset{H_1}{\gtrless}} \frac{l_0 + \frac{MA}{2\sigma^2}}{A/\sigma^2} =: u_0$$

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Thus in the strong signal case, we have a threshold test of the form

$$\phi(\underline{R}) = \begin{cases} 1, & u(\underline{R}) > u_0 \\ 0, & u(\underline{R}) \le u_0 \end{cases}$$

where

$$u(\underline{R}) = \sum_{m=1}^{M} R_m$$

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So we have two integration rules

Linear Detector:

$$u(\underline{R}) = \sum_{m=1}^{M} R_m \underset{H_0}{\overset{H_1}{\gtrless}} u_0$$

Square-Law Detector:

$$T(\underline{R}) = \sum_{m=1}^{M} R_m^2 \underset{H_0}{\overset{H_1}{\gtrless}} T_0$$

In order to characterize the performance of these detectors, we need to find the distributions of $u(\underline{R})$ and $T(\underline{R})$ under both H_0 and H_1.