

Session 23

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Radar Target Detection

ECE678: Radar Engineering
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Nonideal Aspects of Radar Target Detection

- When we derived the matched-filter, we assumed we had complete knowledge of the received signal.
- In most radar detection problems, there is at the very least uncertainty in the phase of the received signal, as round-trip motion of the target through one wavelength corresponds to a 2π change in phase.
- We rarely have *a priori* knowledge of the target position down to a wavelength, so we must consider the effect of this on detection performance.
- In the ideal case of known, constant, received phase and system coherence between pulses, we saw that the matched-filter of a pulse-train corresponded to matched-filtering of the individual pulses and then coherently summing the results. This assumes:
 - knowledge of target-return phase
 - coherence of the radar system from pulse-to-pulse
 - constant target return (in amplitude and phase) from pulse-to-pulse
- In practice, any or all of these may not hold.

- We have seen that optimal detection of a known target in additive white Gaussian noise is provided by the matched filter.
- This corresponds to the following hypothesis testing problem:

$$\begin{array}{lcl}
 H_0 : & r(t) = n(t) & \\
 \text{versus} & & \uparrow \text{white Gaussian noise} \\
 H_1 : & r(t) = s(t) + n(t) & \\
 & \uparrow \text{signal} &
 \end{array}$$

- But two-way motion of one wavelength results in a phase shift of 2π radians in the passband signal (the carrier).
- So in general, we have a complex factor $e^{i\theta}$ applied to the received complex baseband signal.
- Here, θ is unknown, and is often modeled as a random variable uniformly distributed on the interval $[0, 2\pi)$.

- Thus the received signal becomes

$$r(t) = e^{i\theta} s(t) + n(t),$$

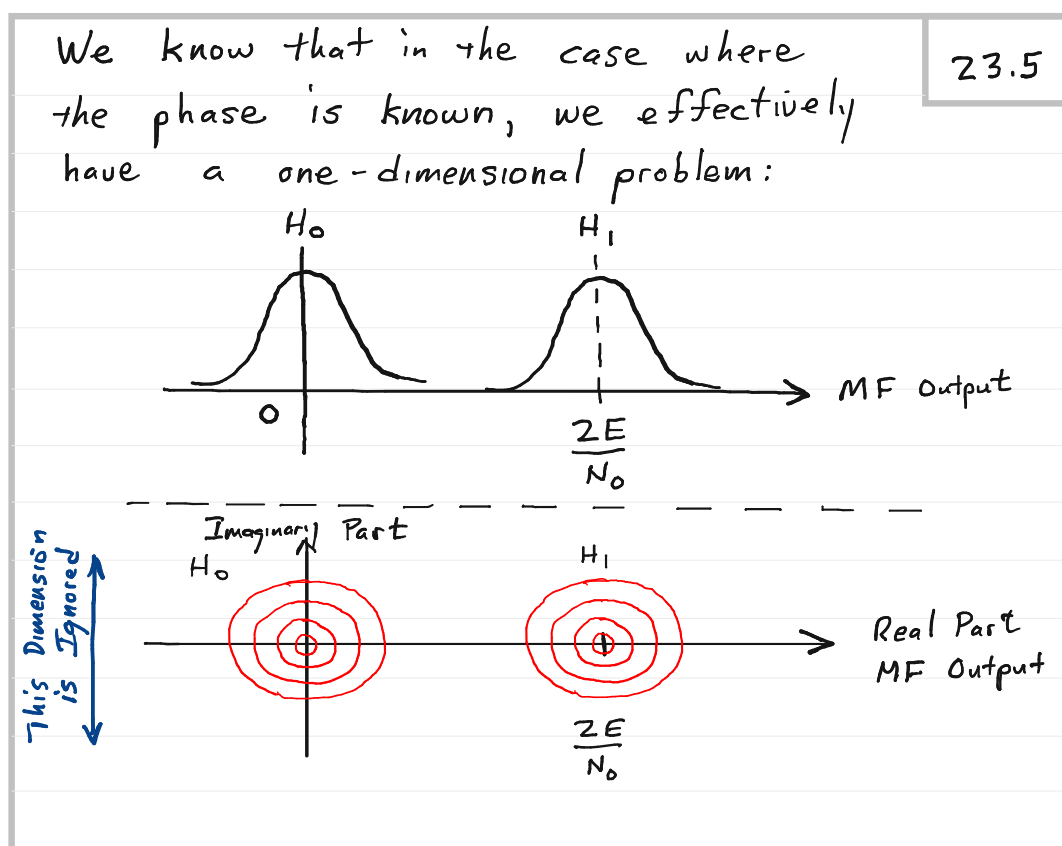
- The new hypothesis testing problem to be considered becomes

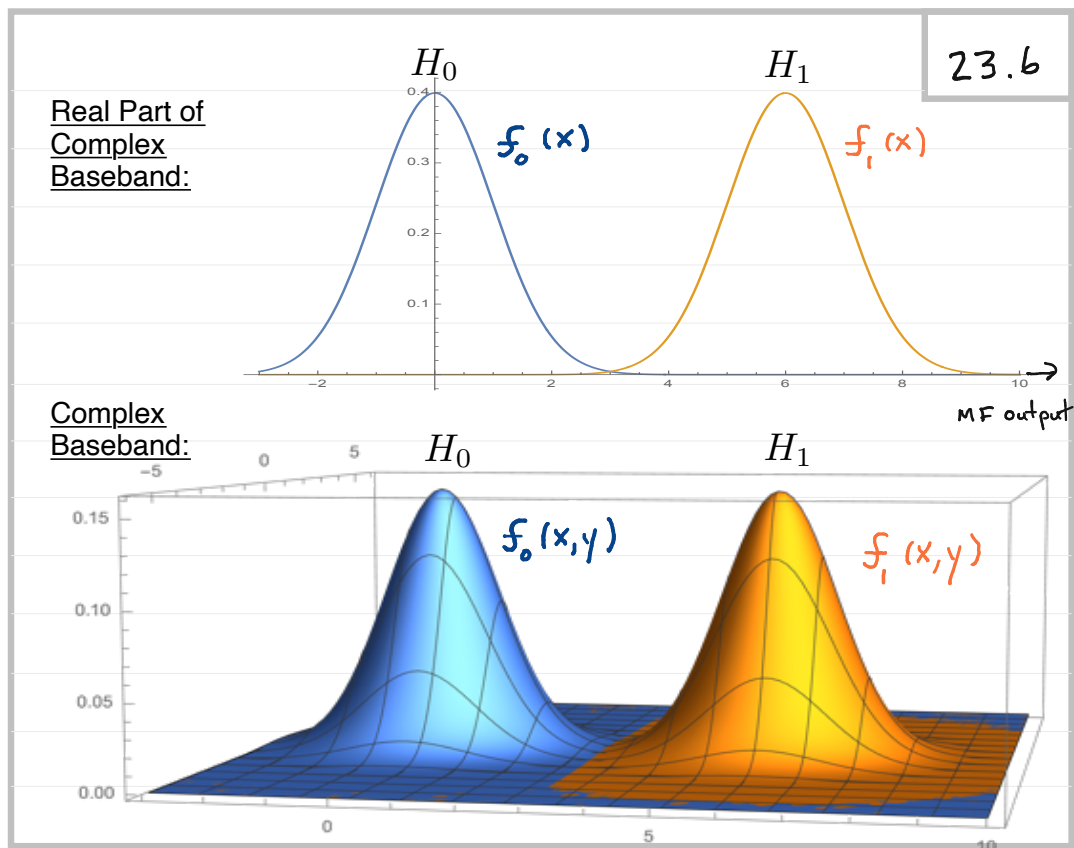
$$\begin{array}{l} H_0 : r(t) = n(t) \\ \text{versus} \\ H_1 : r(t) = e^{i\theta} s(t) + n(t), \end{array}$$

↑ complex white Gaussian noise
↑ signal

where θ is unknown and usually modeled as uniformly distributed on $[0, 2\pi)$.

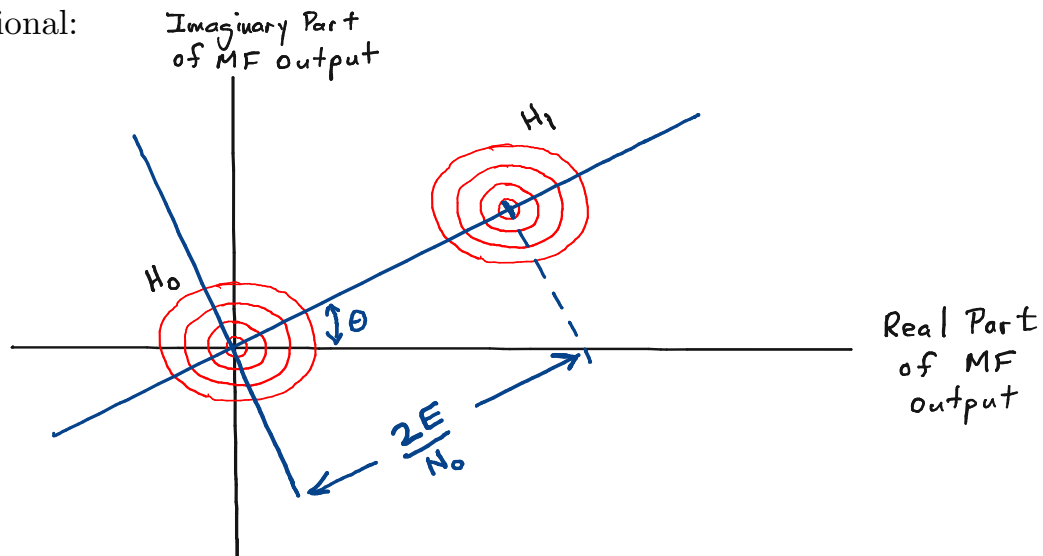
- Also, we assume that $n(t)$ is independent of θ .





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However, in the case where the phase θ is unknown, the problem is intrinsically 2-dimensional:



When we do not know the phase θ , we must consider both dimensions (the real and imaginary part of the matched-filter output.) There is i.i.d. noise in both dimensions, and we get a contribution from both noise components.

The problem of detecting a known signal $s(t)$ with unknown phase θ (i.e., the problem of detecting $e^{i\theta}s(t)$) is called the non coherent pulse detection problem.

Noncoherent Pulse Detection

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In detecting a known pulse with an unknown phase factor $e^{i\theta}$, the detection statistic is the magnitude of the complex matched filter output.

If: $S(t) \rightarrow \boxed{\text{MF}} \xrightarrow[t=T]{} X(T)$

Then: $e^{i\theta} S(t) \rightarrow \underbrace{\boxed{\text{MF}}}_{\text{L.T.I.}} \xrightarrow[t=T]{} \underbrace{i\theta X(T)}_{\text{by linearity of MF.}}$

If we assume $\theta \sim U[0, 2\pi)$, there is no preferred direction for the signal or the noise.

$\Rightarrow$ The phase of the M.F. output is irrelevant.

The magnitude $\sqrt{X_R^2(T) + X_I^2(T)}$ is all that matters as a detection statistic, where

$$X(T) = X_R(T) + i X_I(T)$$

Complex Baseband MF Output

The in-phase (real) and quadrature (imaginary) noise components in the complex baseband matched filter output are i.i.d, zero mean RVs with variance $\sigma^2 = \frac{2E}{N_0}$

They together can be viewed as a complex "circular Gaussian" RV

$$Z = X + iY, \quad X, Y \sim N[0, \sigma^2]$$

$$X \perp Y.$$

Since the magnitude of the noise is given

by $R = \sqrt{X^2 + Y^2}$

and $\theta \sim \tan^{-1}(Y, X)$

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- The output of the matched filter can be shown to have pdf

$$f_{R\Theta}(r, \theta) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \cdot \frac{1}{[\infty, \infty)}(r) \cdot \frac{1}{[0, 2\pi)}(\theta).$$

- Integrating w.r.t. θ , we get

$$f_R(r) = \int_{-\infty}^{\infty} f_{R\Theta}(r, \theta) d\theta = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \cdot \frac{1}{[0, \infty)}(r)$$

pdf under H_0

Rayleigh pdf

Under H_1 (Signal + noise):

The output of the matched filter, for a fixed $\Theta = \theta_0$, can be described by a two-dimensional Gaussian pdf with non-zero means $(\bar{x}, \bar{y})$ given by

$$\bar{x} = \frac{2E}{N_0} \cos \theta_0$$

and
$$\bar{y} = \frac{2E}{N_0} \sin \theta_0.$$

We can write the means in the form

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$$\bar{X} = A \cos \theta,$$

$$\bar{Y} = A \sin \theta.$$

Then we have

$$f_{Re}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x - A \cos \theta)^2}{2\sigma^2}\right\}$$

$$f_{Im}(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y - A \sin \theta)^2}{2\sigma^2}\right\}$$

Then letting

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$$R = \sqrt{X^2 + Y^2}$$

and $\Phi = \tan^{-1}(Y, X),$

we can write the joint pdf of R and Φ as

$$f_{R\Phi}(r, \phi) = \frac{r}{2\pi\sigma^2} \exp\left\{-\frac{(r^2 - 2Ar \cos(\theta - \phi) + A^2)}{2\sigma^2}\right\} \underset{[-\pi, \pi)}{1(\theta)} \underset{[0, \infty)}{1(r)}.$$

Integrating over ϕ , we get

$$f_{IR}(r) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \cdot e^{-A^2/2\sigma^2} \cdot \int_{-\pi}^{\pi} \underbrace{e^{Ar \cos(\theta - \phi)/2\sigma^2}}_{\text{periodic in } \phi} d\phi$$

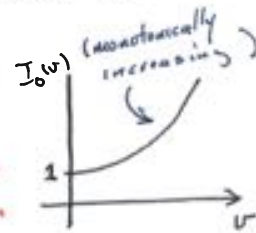
periodic in ϕ .
Integrated over exactly
one period regardless
of value of θ .

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This integral is independent of θ because it extends over exactly one period of $\cos(\theta - \phi)$ regardless of the value of θ . So the value of the integral is independent of θ and can be expressed in terms of the zero-order modified Bessel function

$$I_0(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{r \cos \phi} d\phi = \frac{1}{\pi} \int_0^{\pi} e^{r \cos \phi} d\phi$$

This is how it is often written in tables -



Thus it follows that under H_1 , we can write

$$f_{R,1}(r) = \frac{r}{\sigma^2} \exp \left\{ -\frac{(r^2 + A^2)}{2\sigma^2} \right\} \cdot I_0 \left(\frac{rA}{\sigma^2} \right), r \geq 0$$

where

$$A = \frac{2E}{N_0} \quad = \quad \frac{r}{\mathcal{L}} \exp \left\{ -\frac{(r^2 + \mathcal{L}^2)}{2\mathcal{L}} \right\} I_0(r), r \geq 0$$

and

$$\sigma^2 = \frac{2E}{N_0} \quad \text{where} \quad \mathcal{L} = \frac{2E}{N_0}$$

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Under H_0 :

$$\begin{aligned} f_{R,0}(r) &= \frac{r}{\sigma^2} \exp \left(-\frac{r^2}{2\sigma^2} \right) \cdot 1_{[0,\infty)}(r) \\ &= \frac{r}{\mathcal{L}} \exp \left(-\frac{r^2}{2\mathcal{L}} \right) \cdot 1_{[0,\infty)}(r) \end{aligned}$$

Under H_1 :

$$f_{R,1}(r) = \frac{r}{\mathcal{L}} \exp \left\{ -\frac{(r^2 + \mathcal{L}^2)}{2\mathcal{L}} \right\} I_0(r) \cdot 1_{[0,\infty)}(r)$$

Under H_0 :

$$\begin{aligned} f_{R,0}(r) &= \frac{r}{\sigma^2} \exp \left\{ -\frac{r^2}{2\sigma^2} \right\} \cdot 1_{[0,\infty)}(r) \\ &= \frac{r}{\mathfrak{L}} \exp \left\{ -\frac{r^2}{2\mathfrak{L}} \right\} \cdot 1_{[0,\infty)}(r) \end{aligned}$$

Under H_1 :

$$f_{R,1}(r) = \frac{r}{\mathfrak{L}} \exp \left\{ -\frac{(r^2 + \mathfrak{L}^2)}{2\mathfrak{L}} \right\} I_0(r) \cdot 1_{[0,\infty)}(r)$$

Computing the log-likelihood ratio, we have

$$\ell(r) = \ln \left(\frac{f_{R,1}(r)}{f_{R,0}(r)} \right) = \ln I_0(r) - \frac{\mathfrak{L}^2}{2\mathfrak{L}} \underset{H_0}{\overset{H_1}{\gtrless}} \ln L_0$$

Because

$\ell(r)$ = monotonically increasing function of r

(n.b. $I_0(r)$ is monotonically increasing
 $\ln(\cdot)$ is monotonically increasing
 $\Rightarrow \ln(I_0(r))$ is monotonically increasing.)

the likelihood ratio test reduces to a threshold test on r :

$$\phi(R) = \begin{cases} 1, & R > \lambda_0 \\ 0, & R \leq \lambda_0 \end{cases}$$

The size and power of this test is given by

$$\alpha = \int_{\lambda_0}^{\infty} f_{R,0}(r) dr$$

$$\beta = \int_{\lambda_0}^{\infty} f_{R,1}(r) dr$$

Now if we let $z = \frac{r}{\sigma} = \frac{r}{\sqrt{\lambda}} \Rightarrow r = z\sqrt{\lambda} \Rightarrow dr = \sqrt{\lambda} dz$
and let $\gamma_0 = \frac{\lambda_0}{\sqrt{\lambda}}$, we can write

$$\alpha(\gamma_0) = \int_{\gamma_0}^{\infty} z e^{-z^2/2} dz = e^{-\gamma_0^2/2}$$

from which it follows that the threshold γ_0 yielding a size α test is

$$\gamma_0 = \sqrt{-2 \ln \alpha}$$

Again, with $z = \frac{r}{\sqrt{\lambda}}$ and threshold $\gamma_0 = \frac{\lambda_0}{\sqrt{\lambda}}$ on z , we have

$$\beta(\gamma_0) = \int_{\lambda_0}^{\infty} f_{R,1}(r) dr = \int_{\gamma_0}^{\infty} \frac{r}{\lambda} \exp\left(-\frac{r^2}{2\lambda}\right) \exp\left(-\frac{\lambda}{2}\right) I_0(r) dr$$

$$\text{let } z = \frac{r}{\sqrt{\lambda}} \Rightarrow r = z\sqrt{\lambda} \Rightarrow dr = \sqrt{\lambda} dz$$

$$= \int_{\gamma_0}^{\infty} \frac{z}{\sqrt{\lambda}} \exp\left(-\frac{z^2}{2}\right) \exp\left(-\frac{\lambda}{2}\right) I_0(z\sqrt{\lambda}) \sqrt{\lambda} dz$$

$$= \int_{\gamma_0}^{\infty} z \exp\left\{-\frac{(z^2 + (\sqrt{\lambda})^2)}{2}\right\} I_0(z\sqrt{\lambda}) dz$$

$$= Q(\sqrt{\lambda}, \gamma_0) = Q\left(\sqrt{\frac{2E}{N_0}}, \gamma_0\right)$$

where

$$Q(\alpha, \gamma) \triangleq \int_{\gamma}^{\infty} x \exp\left\{-\frac{(x^2 + \alpha^2)}{2}\right\} I_0(\alpha x) dx$$

= "Marcum Q-function"

So the ROC for the non-coherent pulse detection problem is

$$\beta(\alpha) = Q\left(\sqrt{\frac{2E}{N_0}}, \sqrt{-2 \ln \alpha}\right).$$

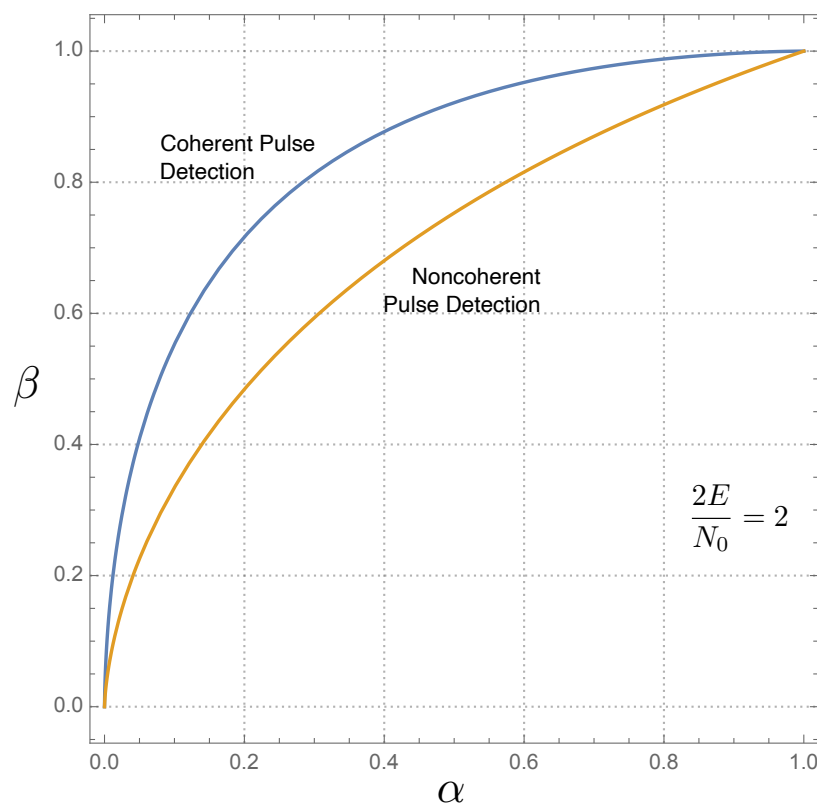
ROC for
Non-coherent
Pulse Detection

The ROC for coherent detection of the same pulse with known phase is

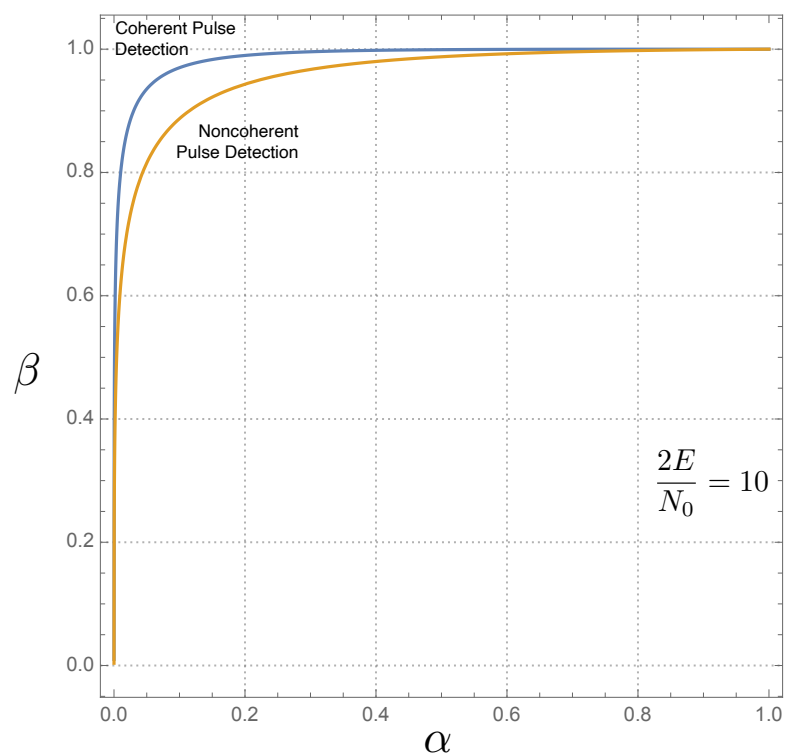
$$\beta(\alpha) = 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \sqrt{\frac{2E}{N_0}}\right)$$

ROC for Coherent
Pulse Detection

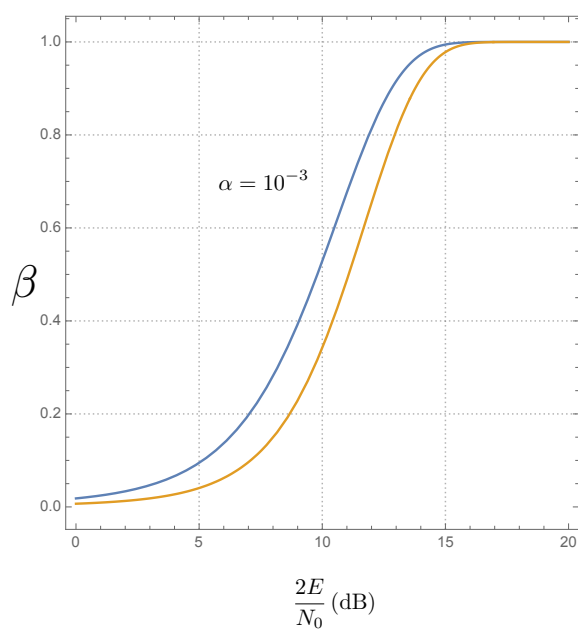
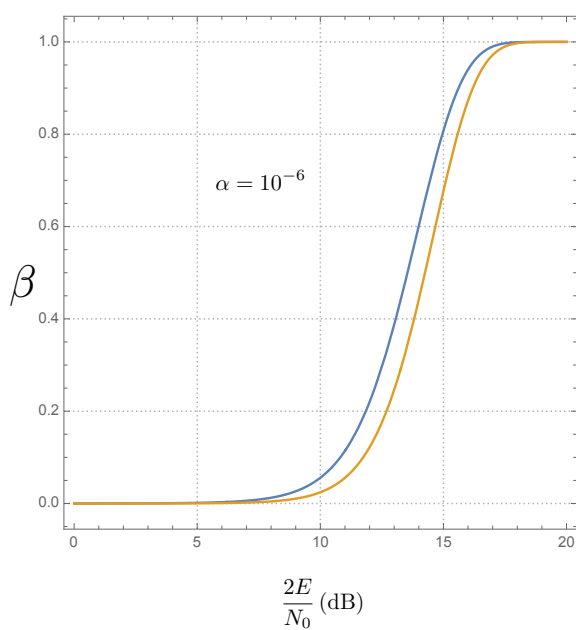
Coherent and Noncoherent Pulse Detection ROCs



Coherent and Noncoherent Pulse Detection ROCs



Probability of Detection as a Function of SNR



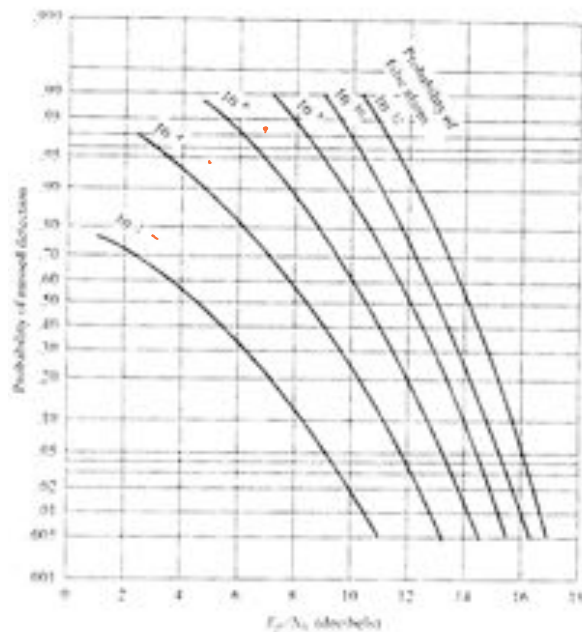


Figure 12.2 Performance of a coherent detector in gaussian noise

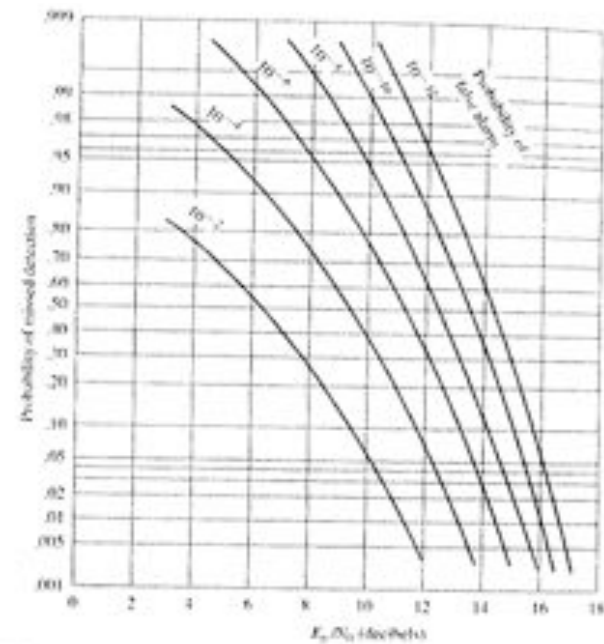
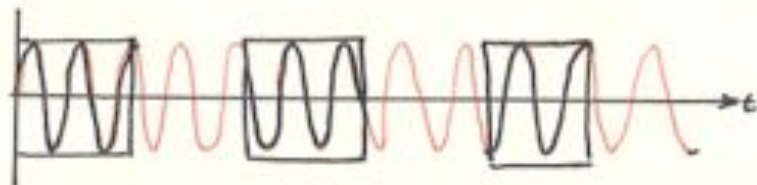


Figure 12.3 Performance of a noncoherent detector in gaussian noise

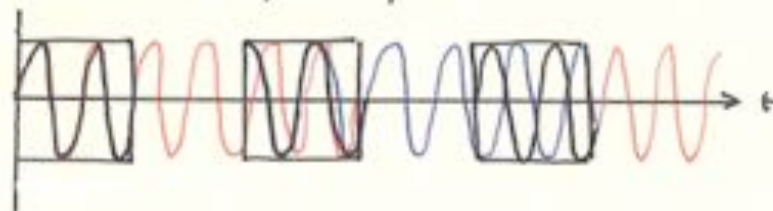
Detection Using Multiple Observations (Pulses)

When we talk about detection of multiple pulses, there are two cases we must consider:

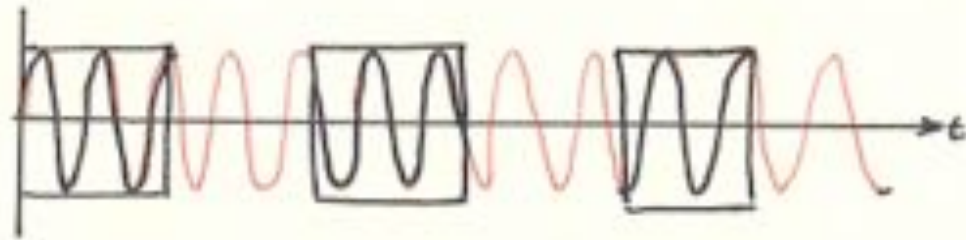
1. A coherent-pulse system



2. A Noncoherent-pulse system

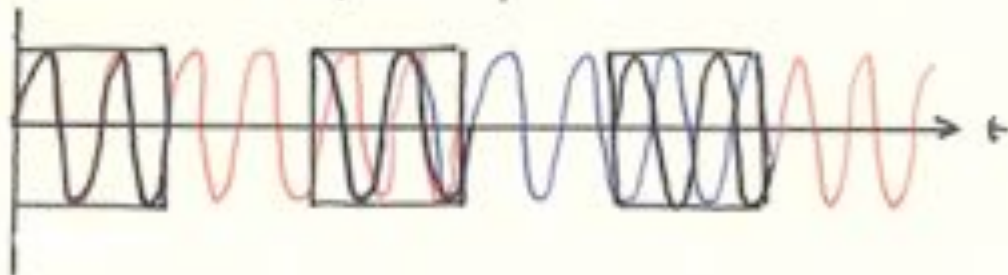


1. A coherent-pulse system



Case 1 in which pulse-to-pulse coherence is maintained is essential for pulse-Doppler radar operation. It allows us to view the phase shift from pulse-to-pulse caused by target motion.

2. A Noncoherent-pulse system



In case 2 - which is used in older and less expensive radar systems, the oscillator is effectively turned on with the transmission of each pulse. This is usually modeled by applying a complex phase factor $e^{j\theta_n}$ to each pulse at baseband, where $\theta_1, \dots, \theta_N$ are modeled as i.i.d $U[0, 2\pi)$.

1 Coherent Multipulse Detection Performance ^{with complete knowledge of received pulses}

We have already looked at this problem when we consider matched-filtering a pulse train

Consider the waveform $s(t)$ constructed by regularly repeating a basic waveform $p(t)$ with amplitude modulation on each pulse:

$$s(t) = \sum_{m=1}^M a_m p(t - (m-1)\Delta), \quad a_m = 1, m=1, \dots, M$$

- Here a_m is the amplitude of the m -th pulse.
- The amplitudes could be binary (e.g., ± 1), real or complex.
- Assume we receive the signal in the presence of zero-mean stationary noise having PSD $S_{nn}(f)$.
- We want to design the *matched filter* to maximize the output SNR at observation time $t = T$.

We know that in this situation, if

$$S(f) = \mathcal{F}\{s(t)\} = \int_{-\infty}^{\infty} s(t) e^{-i2\pi f t} dt,$$

Then the matched filter is given by

$$\hat{H}_M(f) = \frac{S^*(f) e^{-i2\pi f T}}{S_{nn}(f)}.$$

Assume $S_{nn}(f) = N_0/2$

If we take $T = M\Delta$ so that the observation time trailing the last pulse is equal to the observation time between all other pulses, and we note that

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-i2\pi f t} dt = \sum_{m=1}^M a_m P(f) e^{-i2\pi f (m-1)\Delta},$$

where

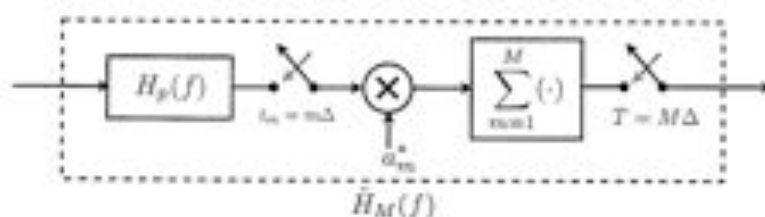
$$P(f) = \int_{-\infty}^{\infty} p(t) e^{-i2\pi f t} dt.$$

So the matched filter can be written as

$$\begin{aligned}\hat{H}_M(f) &= \frac{S^*(f)}{S_{nn}(f)} e^{-i2\pi f M \Delta} \\ &= \sum_{m=1}^M a_m^* \frac{P^*(f)}{S_{nn}(f)} e^{-i2\pi f \Delta} e^{-i2\pi (M-m)f \Delta} \\ &= \sum_{m=1}^M a_m^* H_p(f) e^{-i2\pi (M-m)f \Delta},\end{aligned}$$

where $H_p(f)$ is the single pulse matched filter sampled at time $t = \Delta$ and given by

$$H_p(f) = \frac{P^*(f)}{S_{nn}(f)} e^{-i2\pi f \Delta}.$$



So we see that the matched-filter for the pulse train is obtained by summing the responses for the matched-filter of each pulse:

For a completely known signal

$$H_0: X(T) \sim N\left[0, \frac{2E_s M}{N_0}\right]$$

Because you are adding M independent zero-mean Gaussian RVs with mean 0 and variance $\frac{2E_s}{N_0}$.

$$H_1: X(T) \sim N\left[\frac{2E_s M}{N_0}, \frac{2E_s M}{N_0}\right]$$

Because you are adding M independent Gaussian RVs with mean $\frac{2E_s}{N_0}$ and variance $\frac{2E_s}{N_0}$.

n.b. The standard deviation of the noise grows like $\sqrt{M}$ while the mean grows like M .
SNR increases by $\left(\frac{M}{\sqrt{M}}\right)^2 = \frac{M^2}{M} = M$.

Thus we know that in this case, the ROC is given by

$$\beta(\alpha) = 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \sqrt{\frac{2ME_s}{N_0}}\right)$$

