

Session 22

Order-Statistic CFAR (OS-CFAR)

22.1

Herman Rohling, "Radar CFAR Thresholding in Clutter and Multiple Target Situations," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 19, pp. 608–621, 1983.

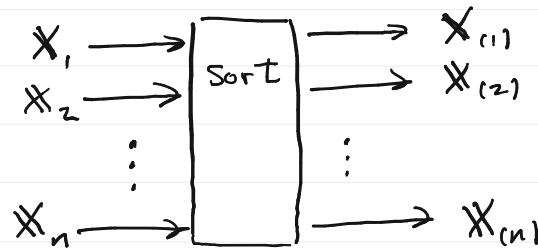
- In OS-CFAR, the average noise power in a region is estimated using an *order statistic*, or ranked sample of the noise power samples in the reference window.
- For example, we might use the *sample median* instead of the *sample mean* to estimate the average noise power.
- While an order statistic estimate is not the maximum likelihood estimate if the samples are independent and statistically homogeneous (i.i.d.), order statistics (e.g., the sample median) are much more robust to deviations from this ideal.

Order Statistics

22.2

Given i.i.d. RVs $X_1, \dots, X_n$, we can order these as

$$X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq \dots \leq X_{(n)}$$



We call this ordering of the RVs $X_1, \dots, X_n$ the order statistics of $X_1, \dots, X_n$.

Suppose $X_1, \dots, X_n$ are i.i.d. RVs with pdf $f_X(x)$ and $F_X(x)$.

22.3

Assume the X_k are absolutely continuous RVs.

What is the p.d.f. of the k -th order statistic $X_{(k)}$?

Let's call the pdf of $X_{(k)}$

$$f_k(x).$$

We want to find $f_k(x)$.

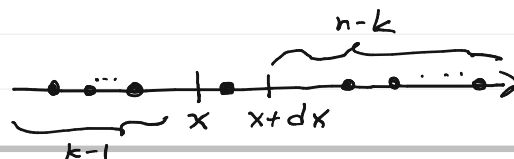
$$f_k(x) dx = P(\{x < X_{(k)} \leq x+dx\}) \quad 22.4$$

a small positive number Δx $= P(B_k)$

where $B_k = \{x < X_{(k)} \leq x+dx\}$.

The event B_k occurs iff

- (i) $k-1$ RVs are less than x
- (ii) one RV is in $[x, x+dx]$
- (iii) The remaining $n-k$ RVs take on values greater than $x+dx$



For any of the n RVs $X_1, \dots, X_n$
(denote the one selected by X)
Define the events

22.5

$$A_1 \triangleq \{X \leq x\}$$

$$A_2 \triangleq \{x < X \leq x+dx\}$$

$$A_3 \triangleq \{X > x+dx\}$$

n.b $\{A_1, A_2, A_3\}$ is a partition of $\mathbb{R}$

For any of the n i.i.d. RVs
 $X_1, \dots, X_n$:

22.6

$$P(A_1) = P(\{X \leq x\}) = F_X(x)$$

$$P(A_2) = P(\{x < X \leq x+dx\}) = f_X(x) \cdot dx$$

$$P(A_3) = P(\{X > x+dx\}) = 1 - F_X(x+dx) \\ \simeq 1 - F_X(x), \quad F_X(x) \text{ is continuous}$$

For the n i.i.d. RVs $X_1, \dots, X_n$, we know that B_k occurs iff

22.7

1. A_1 occurs $k-1$ times,
2. A_2 occurs once,
3. A_3 occurs $n-k$ times.

We can compute the probability of B_k using the multinomial distribution for 3 events.

$$\begin{aligned}
 P(B_k) &= \frac{n!}{(k-1)! 1! (n-k)!} P(A_1)^{k-1} P(A_2)^1 P(A_3)^{n-k} \quad 22.8 \\
 &= \frac{n!}{(k-1)! (n-k)!} [F_*(x)]^{k-1} [f_*(x) dx] [1 - F_*(x)]^{n-k} \\
 &= f_k(x) dx
 \end{aligned}$$

$$\therefore f_k(x) = \frac{n!}{(k-1)! (n-k)!} F_*^{k-1}(x) [1 - F_*(x)]^{n-k} f_*(x)$$

When n is odd, we can set $k = \frac{n+1}{2}$, 22.9
 and we get the order statistic called the sample median $X_{(\frac{n+1}{2})}$ of $X_1, \dots, X_n$.

Equal numbers of RVs lie above and below the sample median.

Order statistics are used in

- (1) Median Filters
- (2) Order Statistic Filters
- (3) OS CFAR Processors

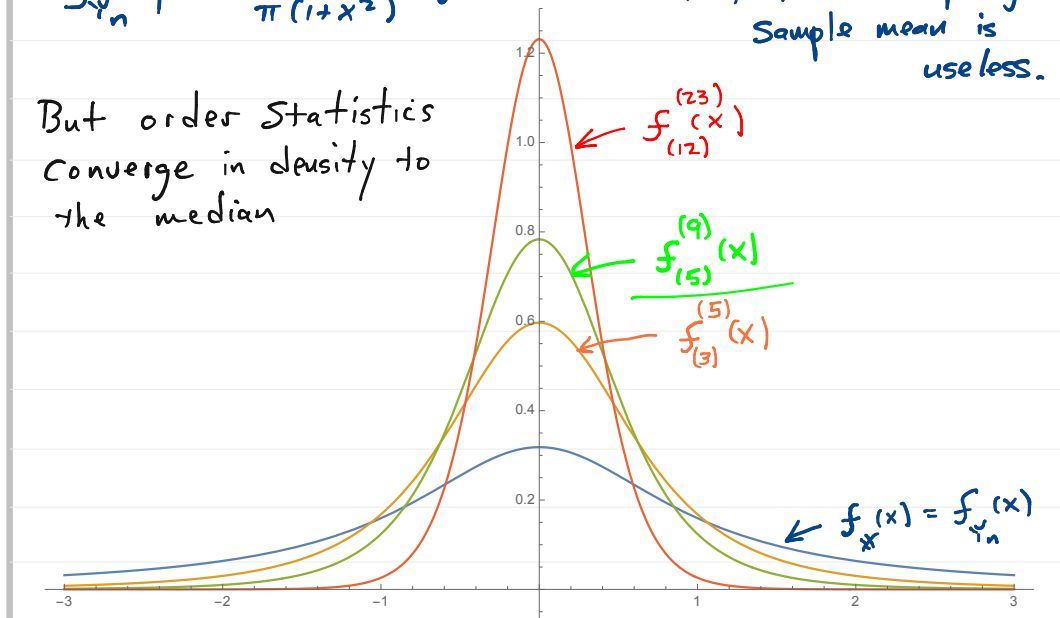
Example: i.i.d Cauchy Random Variables

22.10

$$f_x(x) = f_{Y_n}(x) = \frac{1}{\pi(1+x^2)}, \quad Y_n = \frac{1}{n} \sum_{j=1}^n X_j \leftarrow$$

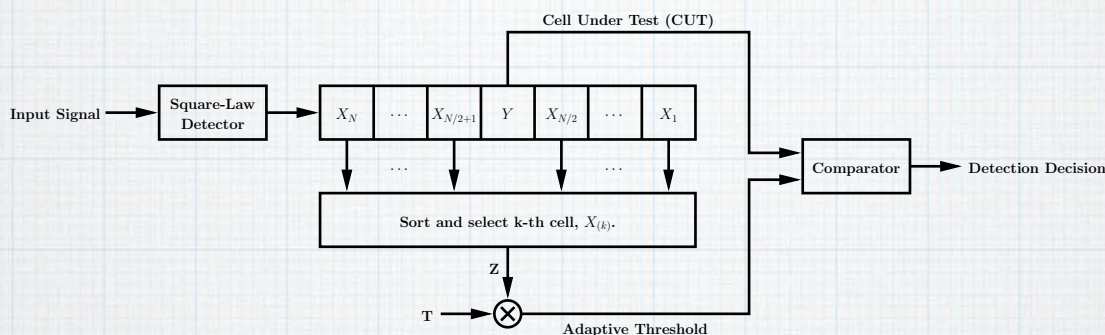
$$f_{Y_n}(y) = \frac{1}{\pi(1+y^2)} \text{ for all } n=1,2,3,\dots \text{ Computing the Sample mean is useless.}$$

But order statistics converge in density to the median



Order-Statistic CFAR (OS-CFAR)

22.11



In OS-CFAR, the reference noise samples $X_1, \dots, X_N$ are sorted from smallest to largest and designated

$$X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(N)}.$$

The k -th order statistic $X_{(k)}$ —or some scaled version of it—can then be used as the mean power estimate.

While the median seems like a logical choice, selecting values of k in the area of $3N/4$ to $4N/5$ have been shown to work well.*

* See: Michael F. Rimbart, *Constant False Alarm Rate Detection Techniques Based on Empirical Distribution Function Statistics*, Ph.D Thesis, School of Electrical and Computer Engineering, Purdue University, August 2005.

Behavior of OS-CFAR

22.12

- OS-CFAR is robust to outliers deviating from a set of homogeneous i.i.d. samples in the reference window because order statistics—especially central order statistics near the median—are robust to outliers.
- This is in fact why statistician John W. Tukey developed and advocated statistical estimation techniques based on them.
- More general results from the theory of *order statistic filters* may also yield interesting new CFAR techniques.
- How well do they behave compared to optimal CA-CFAR when the noise reference samples are i.i.d. ?

Analysis of OS-CFAR

22.13

Assume that $X_1, \dots, X_N$ are i.i.d. samples from a common pdf $f(x)$ having corresponding cdf $F(x)$. If we form the order statistics

$$X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(N)},$$

it can be shown that (See Papoulis, Ch. 8)* the pdf of $X_{(k)}$ is

$$\begin{aligned} f_k(x) &= \frac{n!}{(k-1)!(n-k)!} [F(x)]^{k-1} f(x) [1-F(x)]^{N-k} \\ &= k \binom{n}{k} [F(x)]^{k-1} f(x) [1-F(x)]^{N-k}. \end{aligned}$$

* Also derived on slides 22.2–22.8 of this lecture.

Now if the X_i are i.i.d. with pdf

$$f(x) = \frac{1}{\mu} e^{-x/\mu} \cdot 1_{[0,\infty)}(x),$$

as we have been assuming under H_0 , then this becomes

$$\begin{aligned} f_k(x) &= \frac{n!}{(k-1)!(n-k)!} [1 - e^{-x/\mu}]^{k-1} \cdot \frac{1}{\mu} e^{-x/\mu} \cdot 1_{[0,\infty)}(x) \cdot [e^{-y/\mu}]^{N-k} \cdot 1_{[0,\infty)}(x) \\ &= \frac{k}{\mu} \binom{n}{k} [e^{-x/\mu}]^{N-k+1} [1 - e^{-x/\mu}]^{k-1} \cdot 1_{[0,\infty)}(x). \end{aligned}$$

Thus the pdf of the OS-CFAR statistic $Z = X_{(k)}$ is Equivalently, the p.d.f. of Z is given by

$$f_Z(z) = \frac{k}{\sigma^2} \binom{n}{k} [e^{-z/\mu}]^{N-k+1} [1 - e^{-z/\mu}]^{k-1} \cdot 1_{[0,\infty)}(z).$$

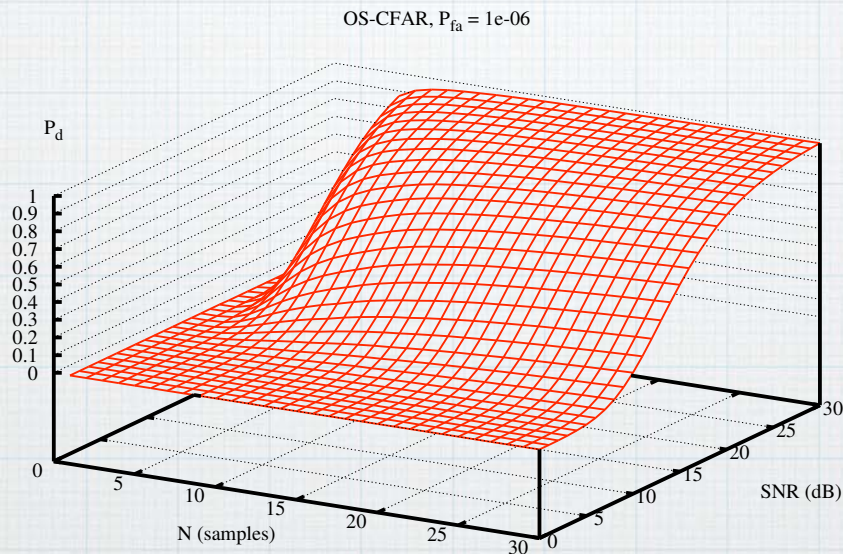
Thus the probability of false alarm for OS-CFAR is given by

$$\begin{aligned} \alpha &= E_Z [P[Y > TZ | H_0]] \\ &= E_Z [\exp\{-TZ/\mu\}] \\ &= \frac{k}{\mu} \binom{N}{k} \int_0^\infty e^{(-T+N-k+1)z/\mu} (1 - e^{-z/\mu})^{k-1} dz \\ &= k \binom{N}{k} \frac{(k-1)!(T+N-k)!}{(T+N)!} \\ &= \prod_{i=0}^{k-1} \left(\frac{N-i}{N-i+T} \right). \end{aligned}$$

Similarly, we can derive the probability of detection as

$$\begin{aligned} P_D &= E_Z [P[Y > TZ | H_1]] \\ &= E_Z [\exp(-TZ/\mu(1+S))] \\ &= \prod_{i=0}^{k-1} \left(\frac{N-i}{N-i+T/(1+S)} \right). \end{aligned}$$

P_D as a function of N and SNR for desired $P_{FA} = 1 \times 10^{-6}$



Mathematica Code to Generate Plots

```
In[1]:=osPfa[n_,k_,t_]:=Module[{i},
  Product[(n-i)/(n-i+t),{i,0,k-1}]]
```

```
In[2]:=osPd[n_,k_,t_,s_]:=Module[{i},
  Product[(n-i)/(n-i+t/(1+s)),{i,0,k-1}]]
```

```
In[3]:=findOS[n_,quantile_]:=Module[{}, Round[n*quantile]]
```

```
In[4]:=findT[n_,k_,pfa_]:=Module[{sol,r},
  sol=FindRoot[osPfa[n,k,r]==0.000001,{r,1}];
  r/.sol]
```

```
In[5]:=caPd[n_,pfa_,s_]:=Module[{}, ((1+s)/(pfa^(-1/n)+s))^n]
```

A Typical Run

22.18

$n=16$

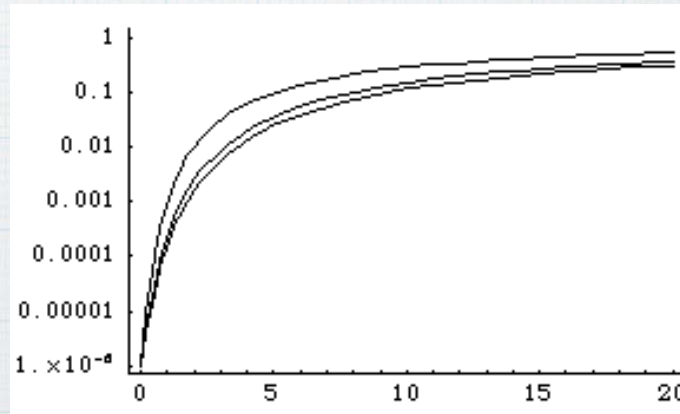
$k=\text{findOS}[n, 4/5]$

$\text{Out}[7]=13$

$t=\text{findT}[n, k, 0.000001]$

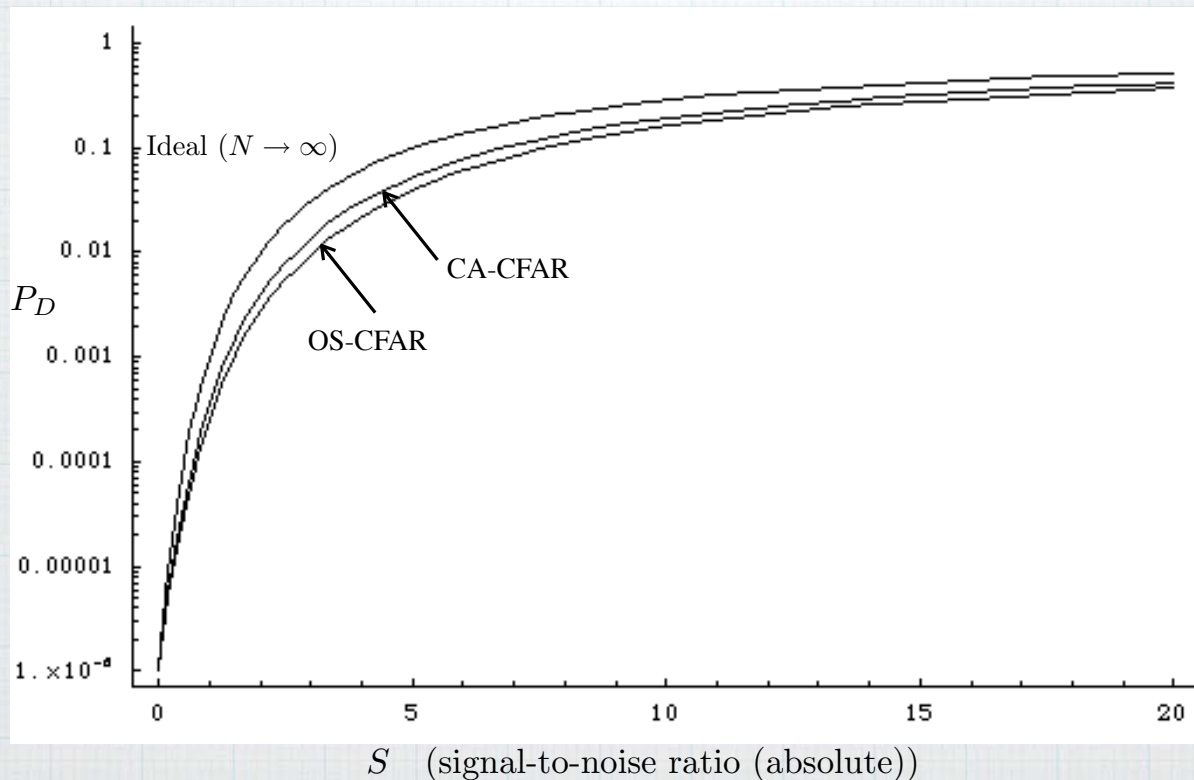
$\text{Out}[8]=16.9527$

$\text{LogPlot}[\{0.000001^{1/(1+s)}, \text{caPd}[n, 0.000001, s], \text{osPd}[n, k, t, s]\}, \{s, 0, 20\}]$

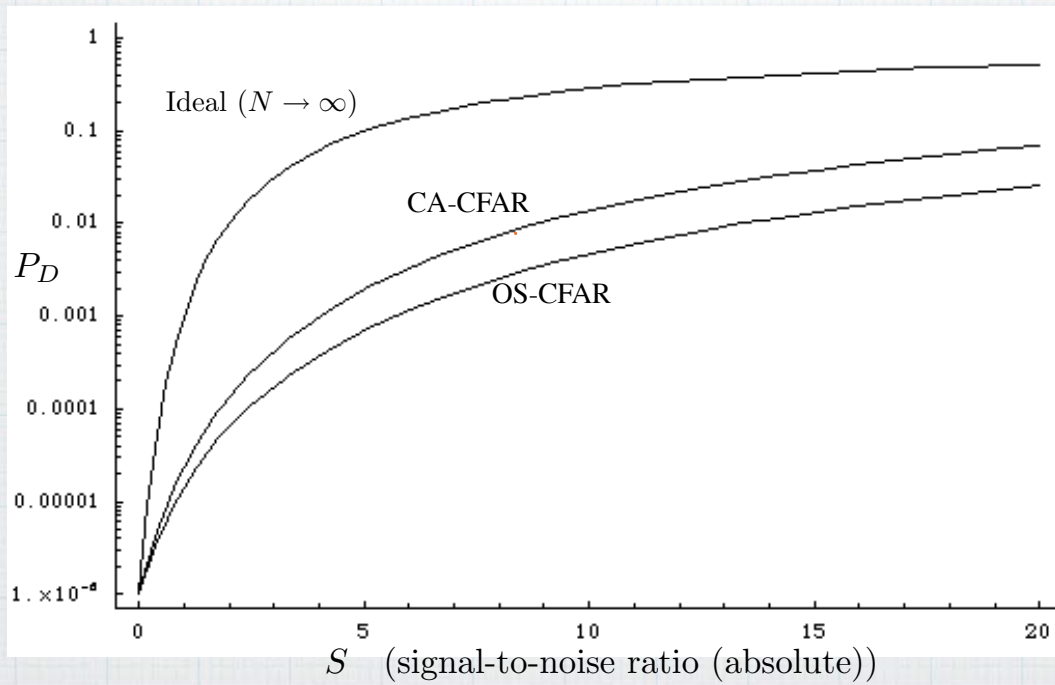


22.19

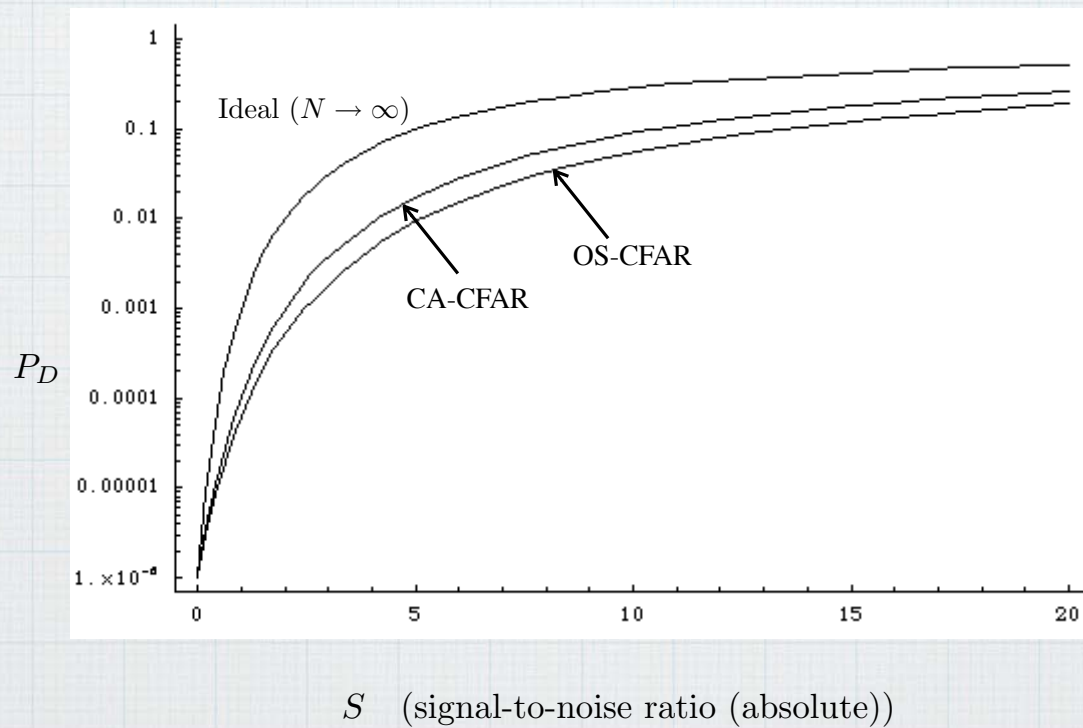
OS-CFAR/CA-CFAR Comparison ($N = 24, k = 19, \alpha = 10^{-6}$)



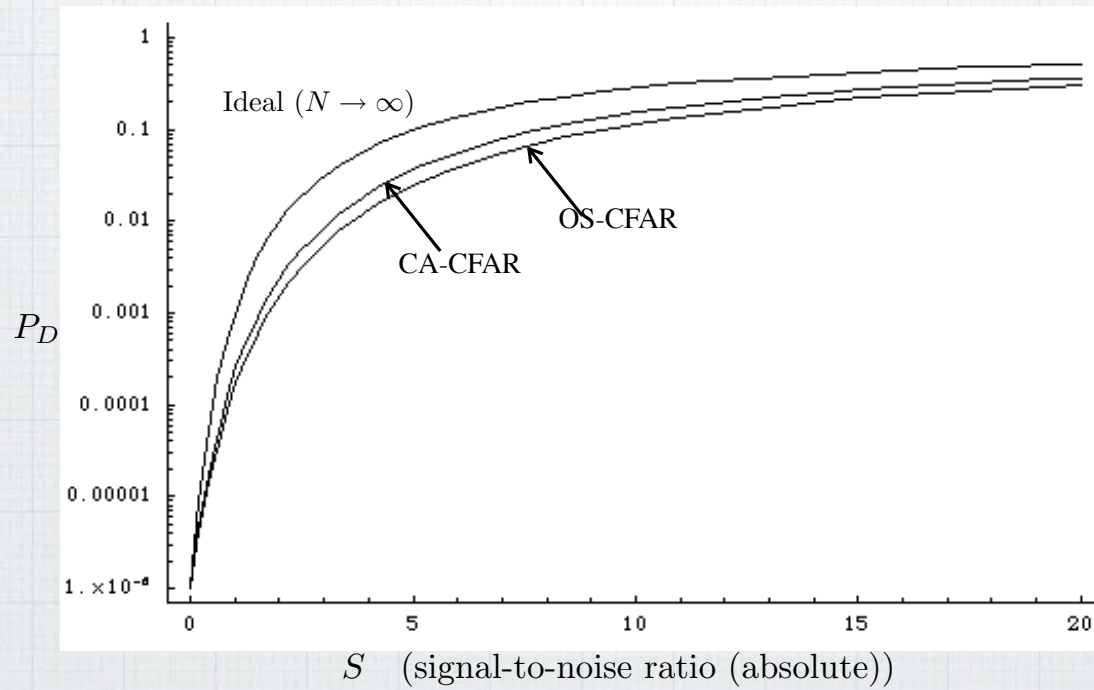
OS-CFAR/CA-CFAR Comparison ($N = 5, k = 4, \alpha = 10^{-6}$)



OS-CFAR/CA-CFAR Comparison ($N = 10, k = 8, \alpha = 10^{-6}$)



OS-CFAR/CA-CFAR Comparison ($N = 16, k = 13, \alpha = 10^{-6}$)



CFAR Processor Performance Comparison

Problem	Processor			
	CA-CFAR	GO-CFAR	SO-CFAR	OS-CFAR
Clutter Edges	Poor	Good	Poor	Good
Interfering Targets	Poor	Poor	Good	Good