Session 18

18.1

The Ambiguity Function of Frequency-Coded Waveforms

The ambiguity function of $s(t) = \sum_{l=0}^{N-1} p(t - lT)e^{-j2\pi\Omega_l t}$ is

$$\chi_s(\tau,\nu) = \chi_s^{(1)}(\tau,\nu) + \chi_s^{(2)}(\tau,\nu),$$

where $\chi_s(\tau,\nu) \triangleq \int^{\infty} s(t)\, s^*(t-\tau)\, e^{+i2\pi\nu t}\, dt$

$$\chi_s^{(1)}(\tau,\nu) = \sum_{m=0}^{N-1} e^{-j2\pi m\nu T} e^{-j2\pi\Omega_m\tau} \chi_p(\tau,\nu),$$

and

$$\chi_s^{(2)}(\tau,\nu) = \sum_{m=0}^{N-1} \sum_{n=0,n\neq m}^{N-1} e^{-j\pi(\Omega_m+\Omega_n)\tau} e^{-j\pi(m+n)T} \cdot\chi_p(\tau+(m-n)T, \nu+(\Omega_n-\Omega_m))$$

n.b. $\beta_s(\tau,\nu) = \chi_s(\tau,-\nu)$.

18.2

The sidelobes are given by

$$\chi_s^{(2)}(\tau,\nu) = \sum_{m=0}^{N-1} \sum_{n=0,n\neq m}^{N-1} e^{-j\pi(\Omega_m+\Omega_n)\tau} e^{-j\pi(m+n)T} \cdot \chi_p(\tau + (m-n)T, \nu + (\Omega_n - \Omega_m))$$

$$\chi_p(\tau + (m-n)T, \nu + (d_n - d_m)/T)$$

Large contribution when *these* equal zero!

$\tau = (n-m)T$ and $\nu = (d_n - d_m)/T)$

or taking $T = 1$ for simplicity...

$\tau = n - m$ and $\nu = d_n - d_m$

18.3

Coincident Sidelobe Approximation

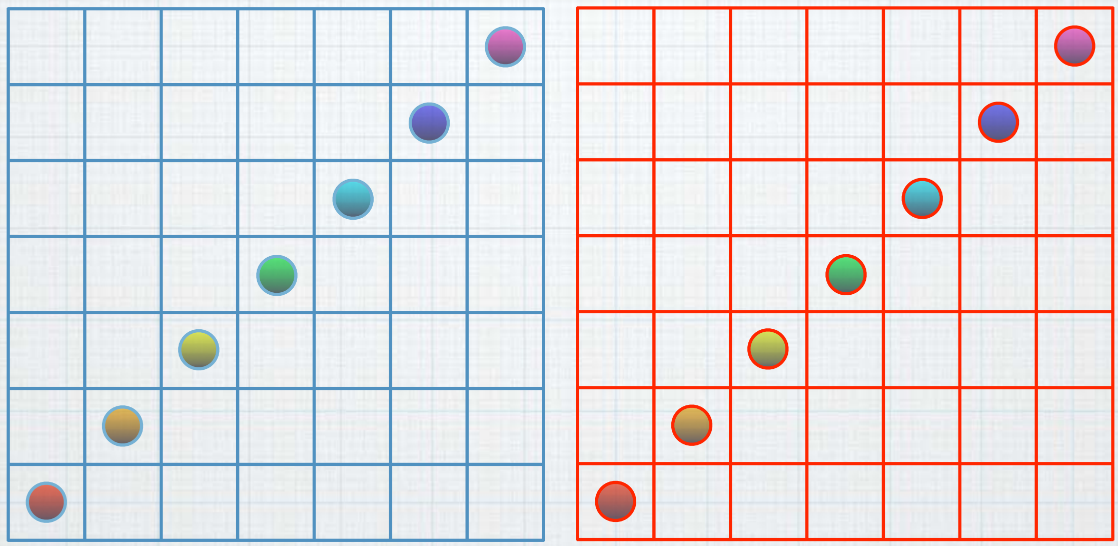
- If we consider only the sidelobe contributions due to the situations where both arguments of the ambiguity function is zero, we want to minimize the number of situations where this occurs.

- We especially want to minimize multiple "hits" for any given delay and Doppler shift.

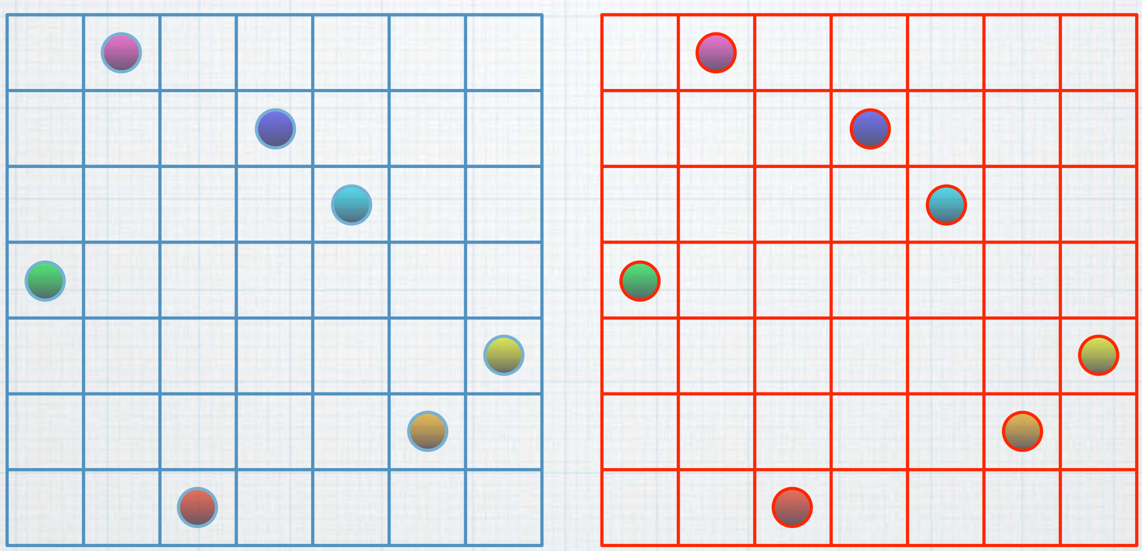
- While this approach only minimizes an approximation of the ambiguity function sidelobes, it is surprisingly effective.

- It is, in fact, the approach John Costas used in designing Costas sequences.

LFM Chirp Sidelobe Overlay Demo



Costas Sidelobe Overlay Demo



18.6

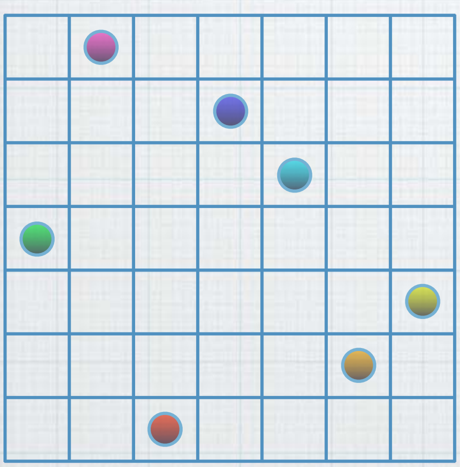
Characteristics of Stepped-Frequency Waveforms

- A wide variety of waveforms with different ambiguity functions can be generated.
- These waveforms can be easily generated and amplified for transmission.
- The ambiguity characteristics of these waveforms can be easily visualized because of their localization in time and frequency.
- Provides a straightforward approach to characterizing "ambiguity state" of a target environment.
- These characteristics make them ideal for adaptive waveform radar.

18.7

The Sidelobe Array

- If we count up the number of sidelobe coincidences for each combination of integer delay-Doppler shifts, we can tabulate the coincidences in an array called the *sidelobe array*.



Doppler ↑

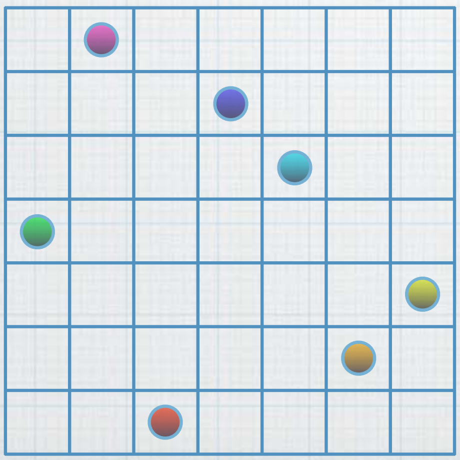
					1							
		1					1					
	1			1				1				
			1	1	1		1					
	1		1	1					1	1		
1				1	1		1		1	1		
						7						
		1	1		1		1	1				1
		1	1					1	1		1	
					1		1	1	1			
				1				1			1	
					1					1		
							1					

Delay →

Costas Sequences

Definition: A **Costas sequence** of length N is a integer frequency firing sequence $\{d_1, \ldots, d_N\}$ (or $\{d_0, \ldots, d_{N-1}\}$ that is a permutation of the integers $1, \ldots, N$ (or $0, \ldots, N-1$) such that the maximum sidelobe height orcoincidence number in the sidelobe array is 1 for any nonzero integer delay-Doppler shift.

An Example ...



					1							
		1					1					
	1			1				1				
			1	1	1		1					
	1		1	1					1	1		
1				1	1		1		1	1		
						7						
		1	1		1		1	1				1
		1	1					1	1		1	
					1		1	1	1			
				1				1			1	
					1					1		
							1					

Reference: N. Levanon and E. Mozeson, ***Radar Signals***, Wiley, 2004 (ISBN 0-471-47378-2)

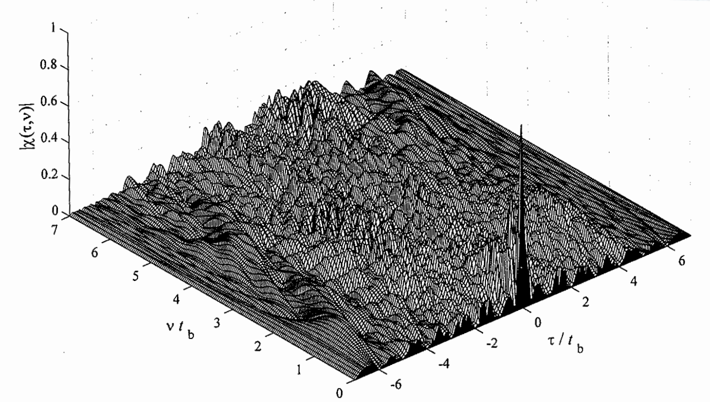


FIGURE 5.4 Partial ambiguity function of a Costas signal with code sequence {4 7 1 6 5 2 3}.

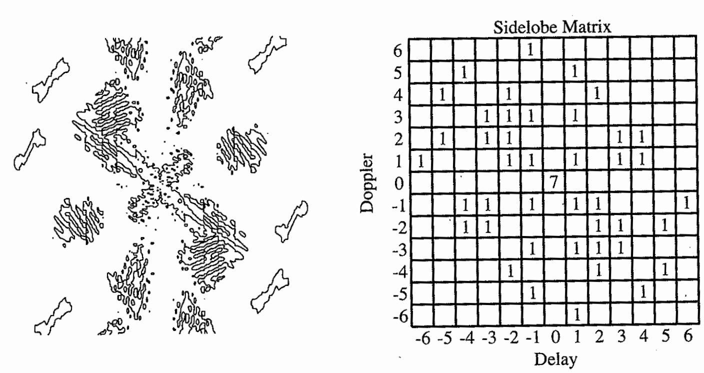


FIGURE 5.5 Ambiguity function contour at 0.125 (left) compared with the sidelobe matrix (right).

- A Length 40 Costas Sequence:

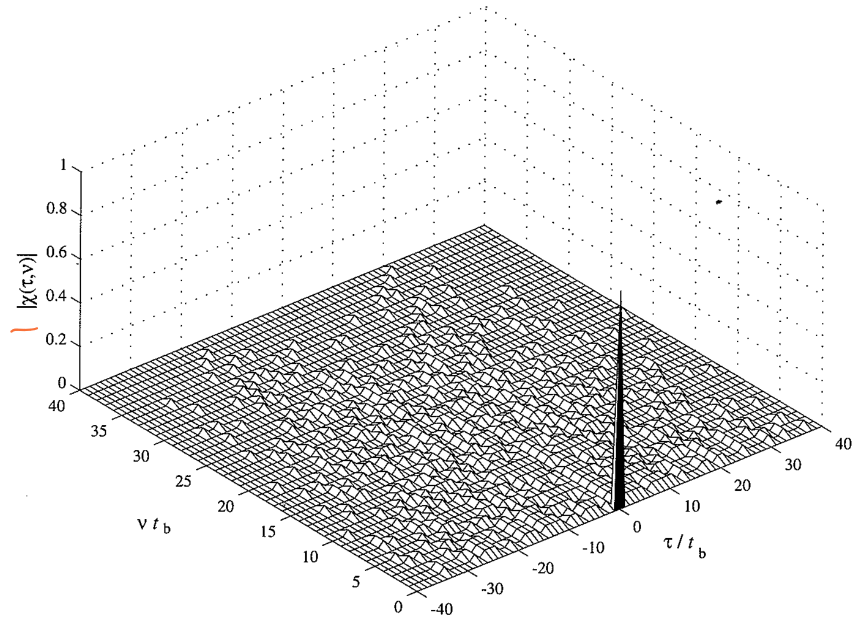


FIGURE 5.9 Ambiguity function of a Costas signal (length $M = 40$) at all relevant grid points.

Reference: N. Levanon and E. Mozeson, ***Radar Signals***, Wiley, 2004 (ISBN 0-471-47378-2)

- A Length 40 Costas Sequence:

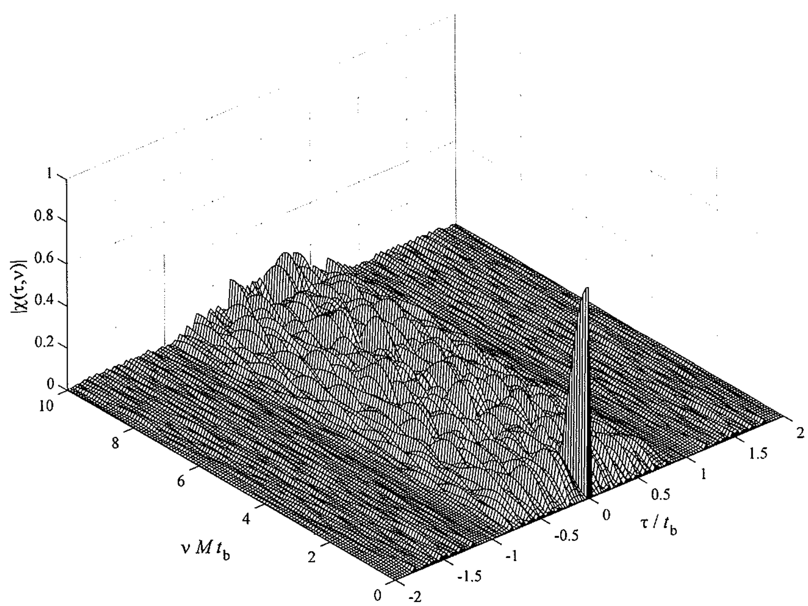


FIGURE 5.10 Ambiguity function of a Costas signal (length $M = 40$) zoom near the origin.

Reference: N. Levanon and E. Mozeson, ***Radar Signals***, Wiley, 2004 (ISBN 0-471-47378-2)

18.12

- A Length 40 Costas Sequence:

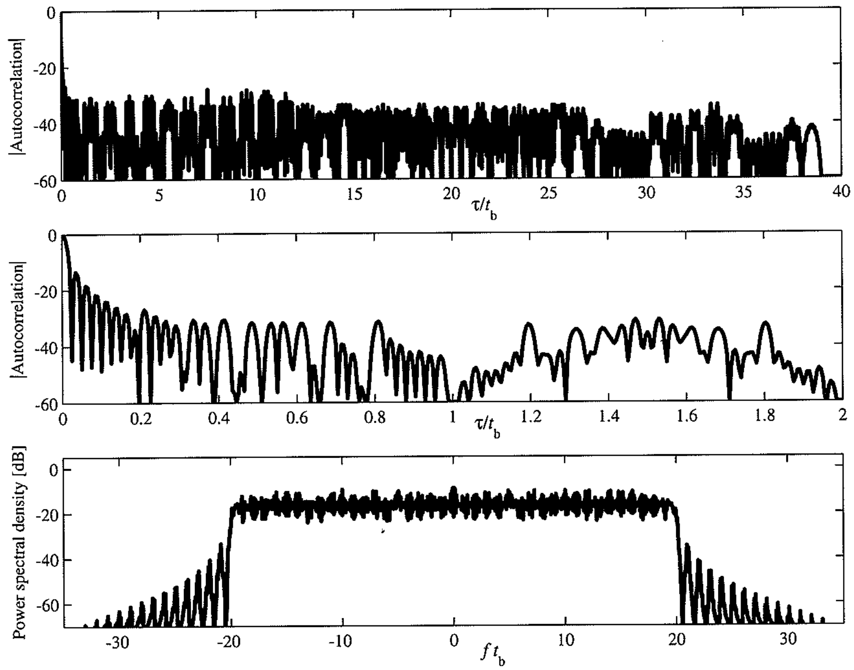


FIGURE 5.11 ACF (top and middle) and the spectrum (bottom) of a Costas signal (length 40).

Reference: N. Levanon and E. Mozeson, ***Radar Signals***, Wiley, 2004 (ISBN 0-471-47378-2)

18.13

Pushing Sequences:

A new class of Frequency-Coded Waveforms for Use in Adaptive Waveform Radar

Chieh-Fu Chang and Mark R. Bell, "Frequency-coded Waveforms for Enhanced Delay-Doppler Resolution," *IEEE Transactions on Information Theory,* vol. 49, no. 11, Nov. 2003, pp. 2960–2971.

18.14

The Ambiguity Function of Frequency-Coded Waveforms

The ambiguity function of $s(t) = \sum_{l=0}^{N-1} p(t - lT)e^{-j2\pi\Omega_l t}$ is

$$\chi_s(\tau, \nu) = \chi_s^{(1)}(\tau, \nu) + \chi_s^{(2)}(\tau, \nu),$$

where

$$\chi_s^{(1)}(\tau, \nu) = \sum_{m=0}^{N-1} e^{-j2\pi m\nu T} e^{-j2\pi\Omega_m \tau} \chi_p(\tau, \nu),$$

and

$$\chi_s^{(2)}(\tau, \nu) = \sum_{m=0}^{N-1} \sum_{n=0, n\neq m}^{N-1} e^{-j\pi(\Omega_m+\Omega_n)\tau} e^{-j\pi(m+n)T} \cdot \chi_p(\tau + (m-n)T, \nu + (\Omega_n - \Omega_m))$$

18.15

Characteristics of Stepped-Frequency Waveforms

- A wide variety of waveforms with different ambiguity functions can be generated.
- These waveforms can be easily generated and amplified for transmission.
- The ambiguity characteristics of these waveforms can be easily visualized because of their localization in time and frequency.
- Provides a straightforward approach to characterizing “ambiguity state” of a target environment.
- These characteristics make them ideal for adaptive waveform radar.

Pushing Sequences

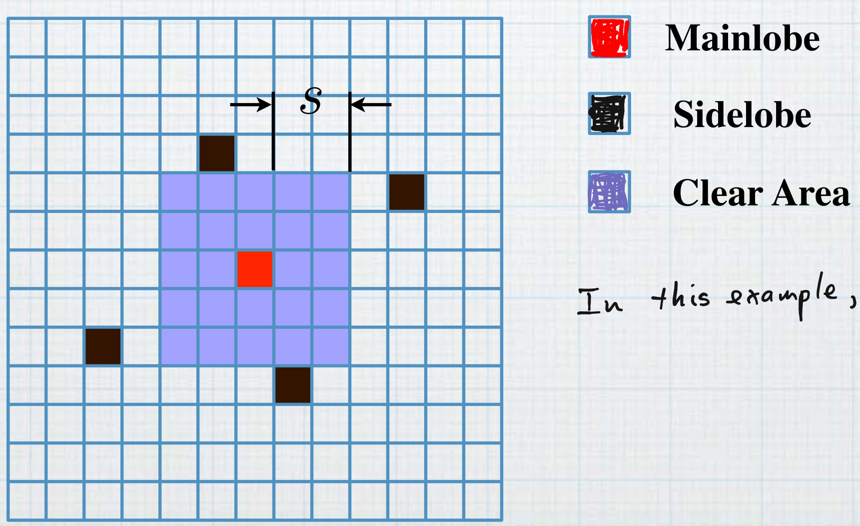
18.16

- Pushing Sequences are frequency coded sequences that have a *clear region* clear of sidelobes surrounding the main lobe.
- *Costas sequences* approximate an ideal thumbtack ambiguity function globally. *Pushing sequences* do so locally.
- Pushing sequences are constructed using much the same intuition that is used for constructing Costas sequences (*difference matrix determination of sidelobes*.)
- Unlike Costas sequences, the frequency sequence need not be a permutation of 1, ... , *N*. Some of the frequencies may left out.
- Arbitrarily large clear areas can be achieved if arbitrarily long sequences are allowed.

Pushing Sequences

18.17

Definition: For the ambiguity function of a signal $s(t)$, a *clear area* of *size* s is a square area centered at the origin of the (τ, ν)-plane, where $|\tau| \leq sT_r$ and $|\nu| \leq s/T$, such that no sidelobe peaks are present in this area.



In this example, s = 2.

Pushing Sequences

18.18

Definition: A sequence having the ambiguity function with a clear area of size s is called a *pushing sequence with power* s, where $s \geq 1$.

Any sequence $\{\underline{d}_N\}$ satisfying either $|i - j| > s$ or $|d_i - d_j| > s$ for all i, j, where $0 \leq i, j \leq N - 1$ and $i \neq j$, will have a clear area of size s and is thus a pushing sequence with power s. This property for a frequency coding sequence is called the *pushing property.*

We are interested in pushing sequences that efficiently fill the geometric array.

Constructing Pushing Sequences

18.19

Lemma: A Costas sequence derived from the Lempel T_4 construction is a pushing sequence of power 1.

Lee codes can be used to construct pushing sequences.

An r-error- correcting Lee code is a length 2 code having close-packed codewords in the geometric representation plane.

The *Lee metric* between codewords must be at least $2r+1$.

Such codes exist for all positive r.

Constructing Pushing Sequences

18.20

Theorem: For every positive integer r, the codewords $\{(k, (2r \oplus 1)k)\}$ form a close-packed r-error correcting dictionary in the Lee metric, where $k = 0, 1, 2...N-1$, $N = 2r^2 + 2r + 1$ and $\oplus$ represents addition modulo N. In that case, the Lee metric between each pair of codewords is at least $2r + 1$.

Theorem: If the hits exist at $(i, (2r \oplus 1)i)$ in the geometric array of $\{\underline{d}_N\}$, where $i = 0, 1, 2...N-1$, $N = 2r^2 + 2r + 1$, r is a positive integer and $\oplus$ represents addition modulo N, then $\{\underline{d}_N\}$ is a pushing sequence with power r.

So the geometric array of a pushing sequence of power *r* is given by the corresponding Lee Code and can be easily constructed.

Sidelobe Locations and Heights

18.21

Theorem: For a Lee pushing sequence with power r, the level of the sidelobe peak at

$$(\tau, \nu) = k_1 V_1 + k_2 V_2,$$

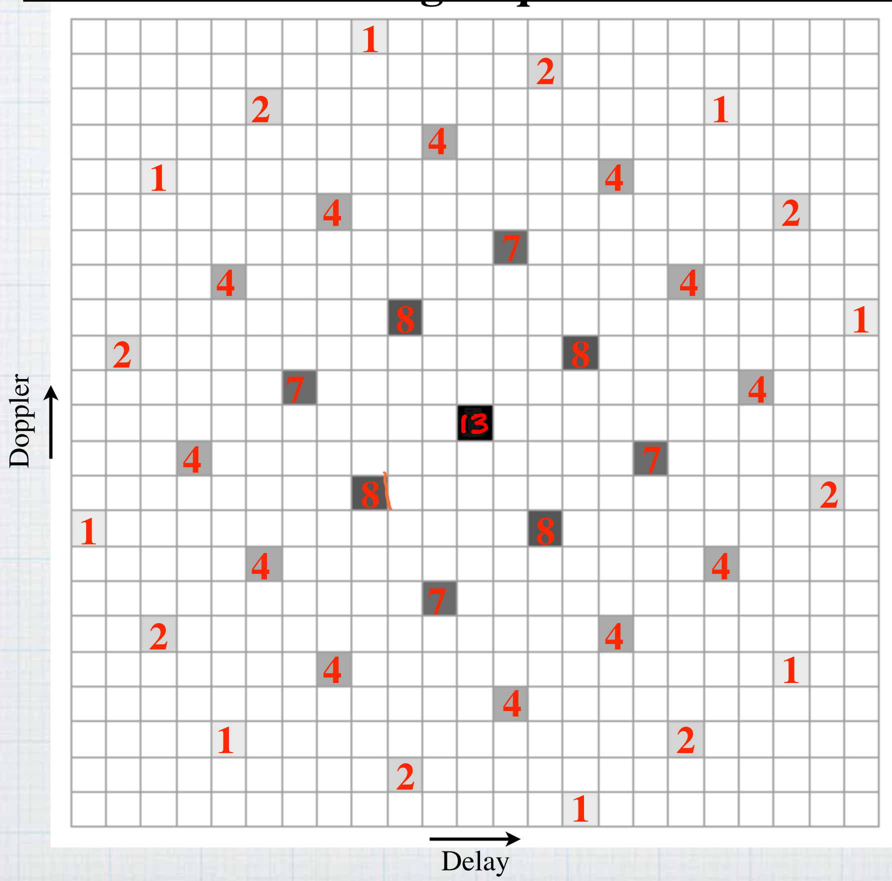
where k_1 and k_2 are integers, $V_1 = (r+1, r)$ and $V_2 = (r, -(r+1))$, is given by

$$l(k_1, k_2) = \left\lfloor \frac{(2r + 1 - |k_1 + k_2|)(2r + 1 - |k_1 - k_2|)}{2} \right\rfloor$$

when $|k_1|, |k_2| \leq (2r - 1)$ and $|k_1| + |k_2| \leq 2r$, and 0 otherwise. Furthermore, these are the only sidelobes.

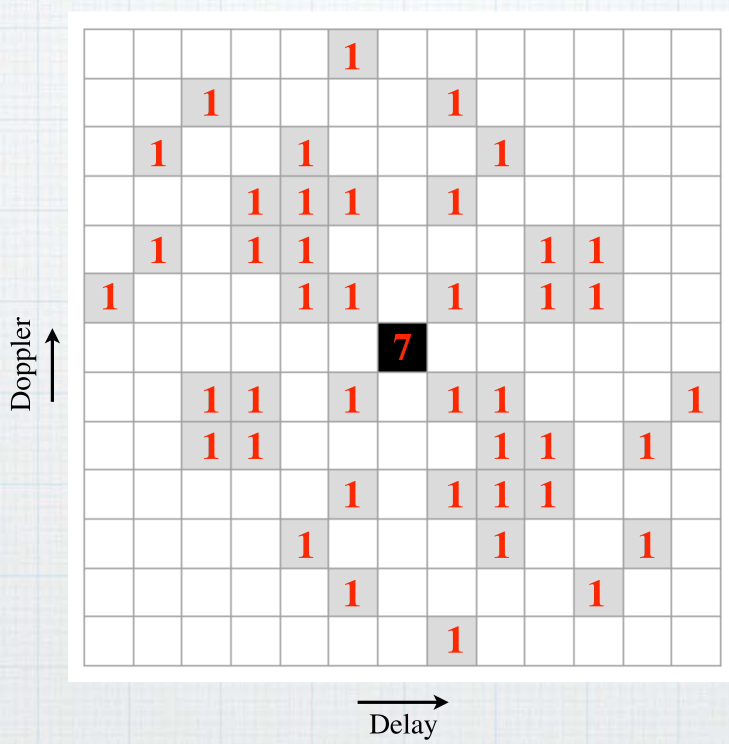
Sidelobes of Pushing Sequence with Power 2

18.22



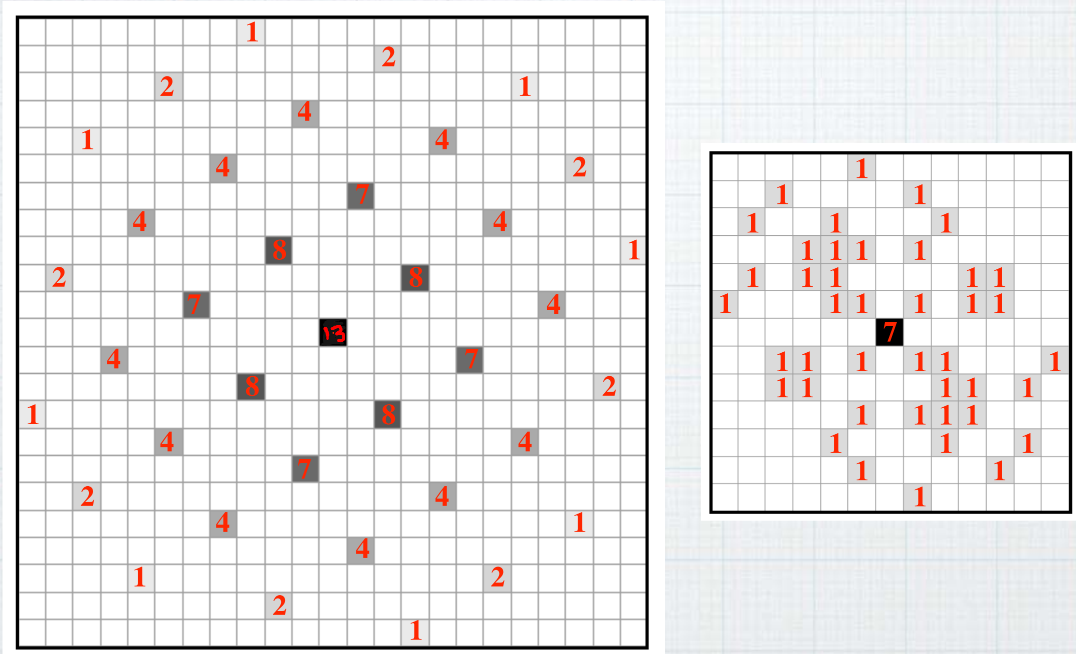
Sidelobes of a Length 7 Costas sequence

18.23



18.24

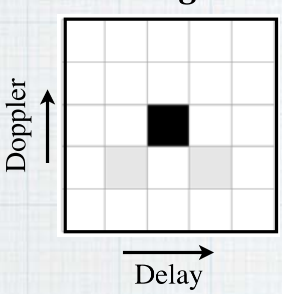
Comparison of *Pushing* and *Costas* Sidelobe Matrices



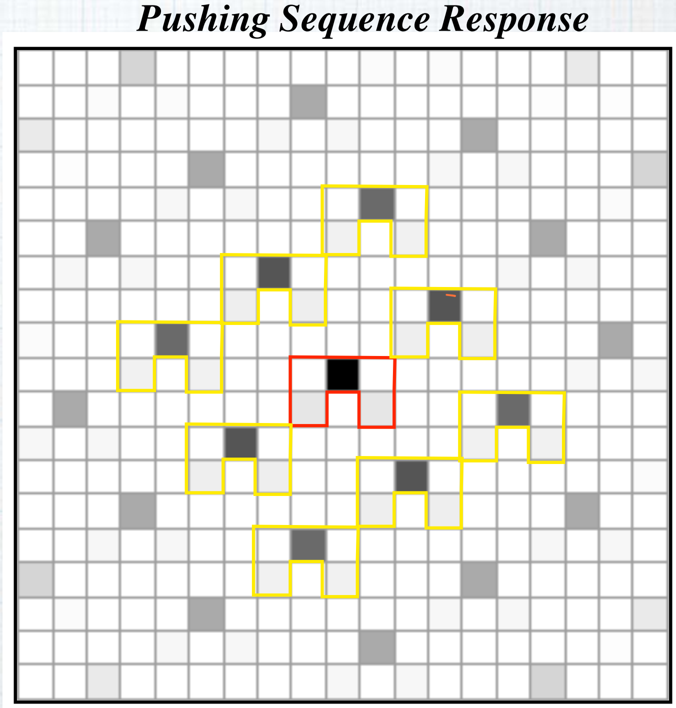
18.25

Waveform Response to a Target

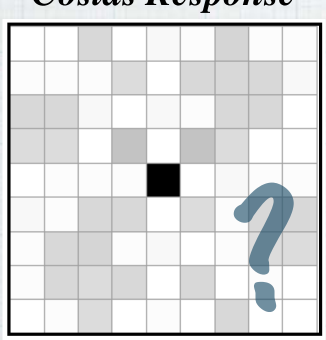
Target



Pushing Sequence Response

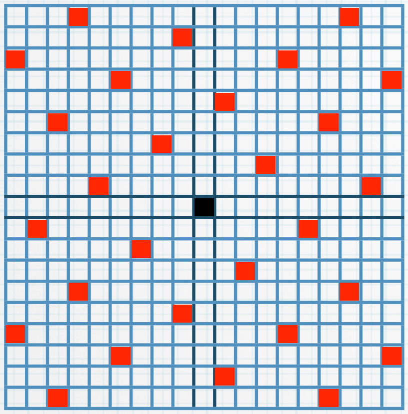


Costas Response

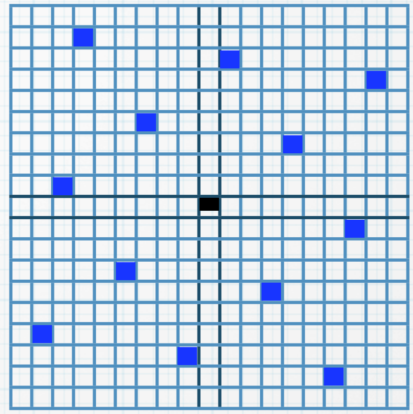


Of course you must know that there are no targets present at another sidelobe location

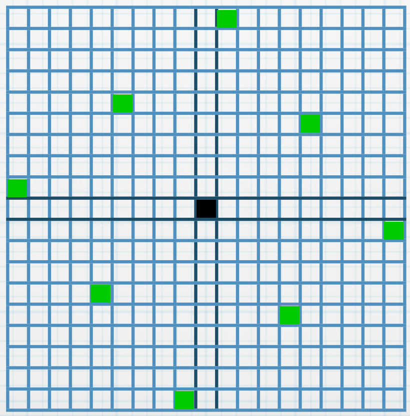
Sidelobe Distributions Available for Power r = 2, 3 and 4



$r = 2$

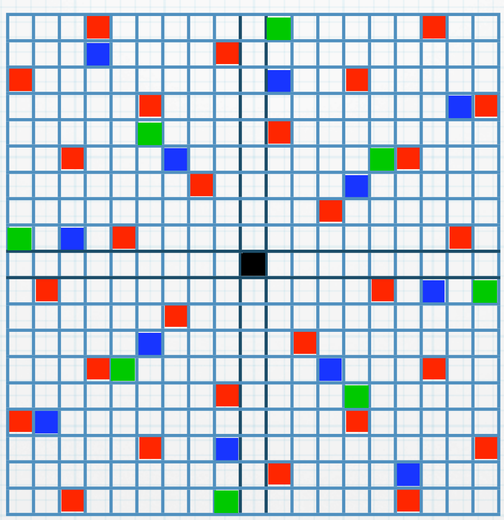


$r = 3$



$r = 4$

Sidelobe Distributions Available for Power r = 2, 3 and 4



$r = 2$
$r = 3$
$r = 4$

18.28

Sequence Length N Increases with Power r

$$N = 2r^2 + 2r + 1$$

r	N
2	13
3	25
4	41
5	61
6	85

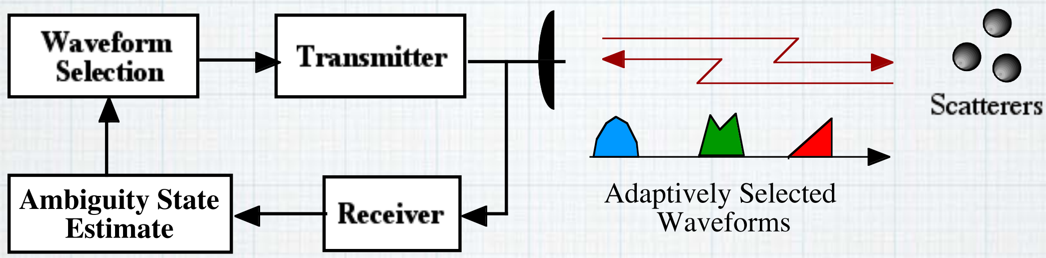
Increasing r significantly increases N, and hence the total waveform duration, total bandwidth, and time bandwidth-product.

18.29

Summary of Pushing Sequence Characteristics

- Frequency coded waveforms that are easy to generate and process.
- Mainlobe of ambiguity function surounded by clear area of arbitrarily large size (power)
- Large sidelobes located on a regular lattice outside of clear region.
- Sidelobe locations and sizes completely determinable.
- Need to know targets are not present at sidelobes.

Adaptive Waveform Radar



- We need waveforms that are easy to generate and transmit.
- We need waveforms for which we can estimate ambiguity functions and the inverse problem of ambiguity state.
- Frequency-coded waveforms seem like a good choice.

Adaptive Waveform Radar

1. Transmit *frequency coded stepped chirp* for Doppler tolerant detection.
2. If a target is detected, transmit a *Costas sequence* for a high-resolution delay-Doppler measurement.
3. If higher resolution is required for small targets that may be masked is Costas sidelobes, transmit *pushing sequence*

   - Target locations from Costas sequence measurement are needed for appropriate *pushing sequence* selection.

Adaptive Waveform Radar