

Session 16

16.1

Cross-Ambiguity Functions (Cont.)

In radar and sonar, one of the functions in $\beta_{rs}(\tau, \nu)$ is often known, while the other one is a random process:

Transmit: $s(t) \Rightarrow \text{AAF}\{s(t)\} = \beta_s(\tau, \nu).$

Receive: $r(t) = a \cdot s(t - \tau_0)e^{i2\pi\nu_0 t} + n(t).$

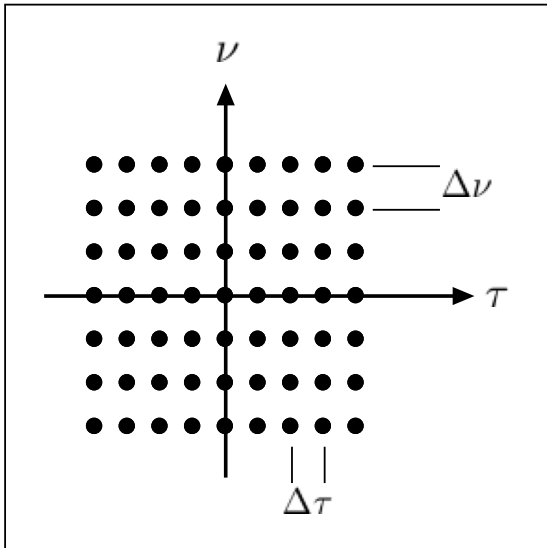
The cross-ambiguity function between $r(t)$ and $s(t)$ is

$$\begin{aligned}
 \beta_{rs}(\tau, \nu) &= \int_{-\infty}^{\infty} r(t) s^*(t - \tau) e^{-i2\pi\nu t} dt \\
 &= \int_{-\infty}^{\infty} [a s(t - \tau_0) e^{-i2\pi\nu_0 t} + n(t)] s^*(t - \tau) e^{-i2\pi\nu t} dt \\
 &= a \int_{-\infty}^{\infty} s(x) s^*(x - [\tau - \tau_0]) e^{-i2\pi(\nu - \nu_0)(x + \tau_0)} dx \\
 &\quad + \int_{-\infty}^{\infty} n(t) s^*(t - \tau) e^{-i2\pi\nu t} dt \\
 &= a e^{-i2\pi(\nu - \nu_0)\tau_0} \cdot \underset{\substack{\uparrow \\ \text{ambiguity function Response}}}{\beta_s(\tau - \tau_0, \nu - \nu_0)} + \underset{\substack{\uparrow \\ \text{zero-mean noise term}}}{\mathcal{M}(\tau, \nu)}.
 \end{aligned}$$

Cross-Ambiguity Functions (Cont.)

What most real radars do is compute $\beta_{rs}(\tau, \nu)$ on a grid of (τ, ν) values using a bank of matched filters:

$$\{(\tau_m, \nu_n) : \tau_m = m\Delta\tau, \nu_n = n\Delta\nu; m, n \text{ integers}\}$$

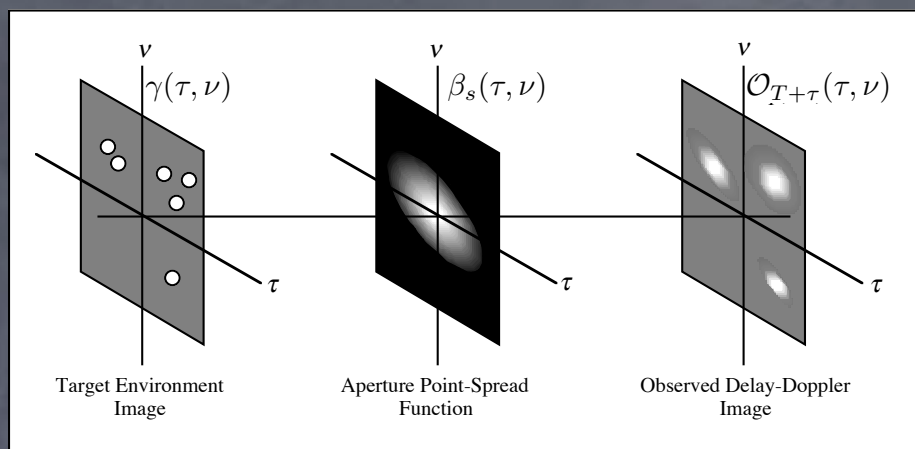


For $\tau_m \approx \tau_0$ and $\nu_n \approx \nu_0$,
the response is approximately

$$\beta_{rs}(\tau_m, \nu_n) \approx aE_s + \mathcal{M}(\tau_m, \nu_n).$$

$\beta_s(0,0) \approx E_s$
 $\downarrow$

By sampling finely enough, the system generates an “image” in the coordinates (τ, ν) .



16.3

- 👁 If we think of a radar as an imaging system, then the ambiguity function behaves much like the point-spread function or impulse response of the system.
- 👁 High-resolution images require sharp ambiguity functions $\Rightarrow$ waveform design prob.

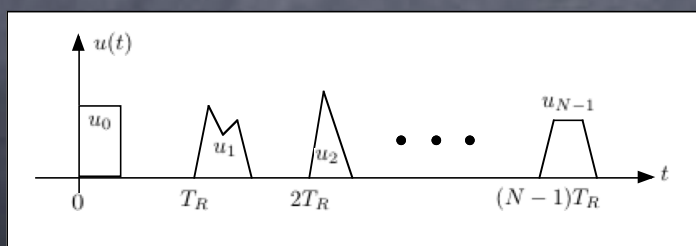
The Ambiguity Function of a Pulse Train

Consider a passband signal

$$s(t) = \text{Re}\{u(t)e^{i2\pi f_c t}\},$$

where the complex baseband signal $u(t)$ is of the form

$$u(t) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} u_n(t - nT_R),$$



The Ambiguity Function of a Pulse Train (Cont.)

The asymmetric ambiguity function of $u(t)$ is

$$\begin{aligned} \beta_u(\tau, \nu) &= \int_{-\infty}^{\infty} u(t)u^*(t - \tau) e^{-i2\pi\nu t} dt \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} u_n(t - nT_R) \right] \left[\frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} u_m(t - mT_R - \tau) \right]^* e^{-i2\pi\nu t} dt \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \int_{-\infty}^{\infty} u_n(t - nT_R) u_m^*(t - mT_R - \tau) e^{-i2\pi\nu t} dt \\ &= \frac{1}{N} e^{-i2\pi\nu NT_R} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \int_{-\infty}^{\infty} u_n(x) u_m^*(x - [\tau - (n - m)T_R]) e^{-i2\pi\nu t} dt \\ &= \frac{1}{N} e^{-i2\pi\nu NT_R} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \beta_{n,m}(\tau - (n - m)T_R, \nu). \end{aligned}$$

where

$$\beta_{n,m}(\tau, \nu) = \int_{-\infty}^{\infty} u_n(t) u_m^*(t - \tau) e^{-i2\pi\nu t} dt.$$

If all of the $u_n(t)$ are identical:

$$u_n(t) = c(t), \quad n = 0, 1, 2, \dots, N-1,$$

then

$$\beta_u(\tau, \nu) = \frac{1}{N} \sum_{p=-(N-1)}^{N-1} e^{-i\pi\nu(N-1+p)T_R} \cdot \beta_c(\tau - pT_R, \nu) \frac{\sin [\pi\nu(N - |p|)T_R]}{\pi\nu(N - |p|)T_R}.$$

If the duration of the pulse $t_p < T_R/2$, the pulses do not overlap in the ambiguity function, and $|\beta_u(\tau, \nu)|$ simplifies to

$$|\beta_u(\tau, \nu)| = \frac{1}{N} \sum_{p=-(N-1)}^{N-1} |\beta_c(\tau - pT_R, \nu)| \cdot \left| \frac{\sin [\pi\nu(N - |p|)T_R]}{\pi\nu(N - |p|)T_R} \right|.$$

If we take $c(t)$ to be a unit energy rectangular pulse of duration t_p :

$$c(t) = \frac{1}{t_p} \cdot 1_{[0, t_p]}(t), \quad \text{where } t_p < T_R/2,$$

a plot of $|\beta_u(\tau, \nu)|$ appears as follows:

Ambiguity function
of a pulse-train
made up of 5
rectangular pulses

$$|\beta_u(\tau, \nu)| = \frac{1}{N} \sum_{p=-(N-1)}^{N-1} |\beta_c(\tau - pT_R, \nu)| \cdot \left| \frac{\sin [\pi\nu(N - |p|)T_R]}{\pi\nu(N - |p|)T_R} \right|$$

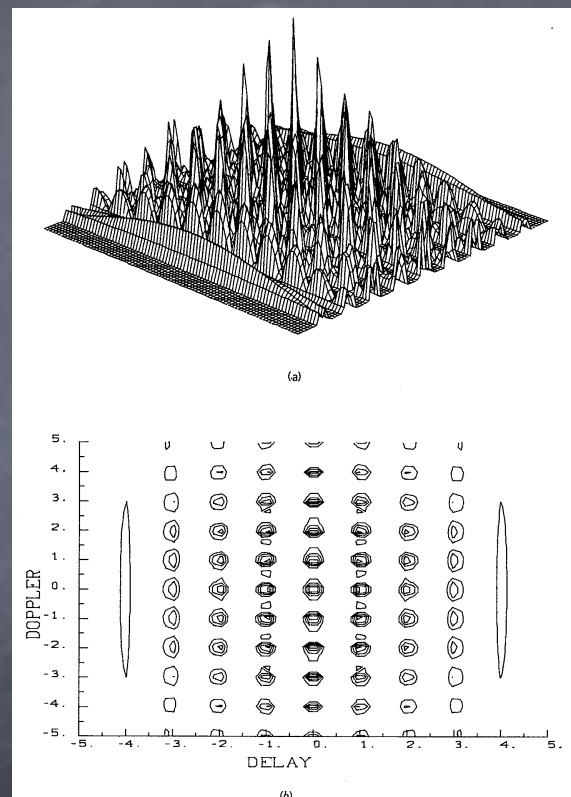


Figure 7.7 The ambiguity function of five coherent pulses ($T_R = 1$, $t_p = 0.2$): (a) 3-D view. (b) Contour plot.

Processing a Coherent Pulse Train

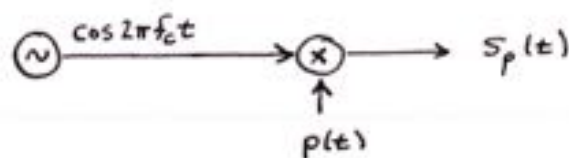
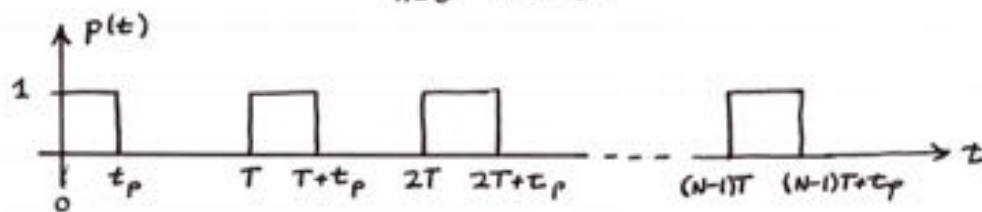
16.8

Assume a radar transmits a passband signal of the form

$$S_p(t) = p(t) \cos 2\pi f_c t$$

where

$$p(t) = \sum_{n=0}^{N-1} 1_{[0, t_p]}(t - nT).$$



The signal reflected back from a point-target with radial velocity $\dot{R}$ and received by the radar is of the form

16.9

$$r(t) = p(t) \cos [2\pi f_c t + \phi(t)].$$

(n.b. we have re-oriented the time origin to account for the round-trip delay)

Here, $\phi(t)$ is due to the Doppler shift:

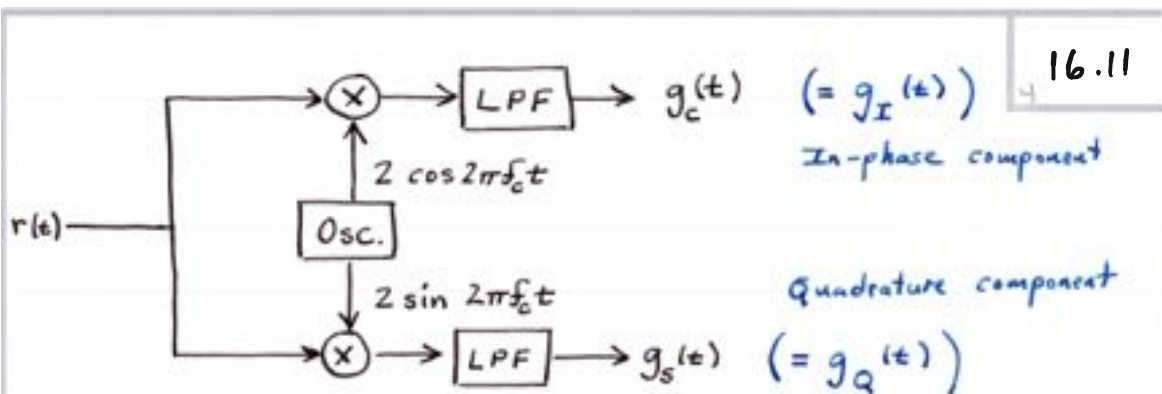
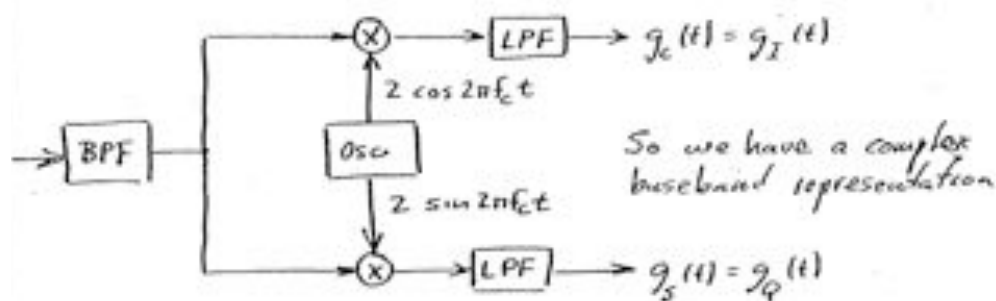
$$\phi(t) = +2\pi f_0 t = -2\pi \frac{2\dot{R}}{\lambda} t.$$

n.b. We are assuming $p(t)$ is narrowband w.r.t. carrier f_c and that time dilation/compression of $p(t)$ is negligible.
Narrowband Assumption

We would like to find the complex baseband representation of the received signal $r(t)$ via processing:

16.10

- Pass the signal through a bandpass filter that passes virtually all of the signal, but little outside of the signal's frequency band.
- Use an I/Q demodulator to get the in-phase and quadrature components.



$$g_c(t) = p(t) \cos \phi(t)$$

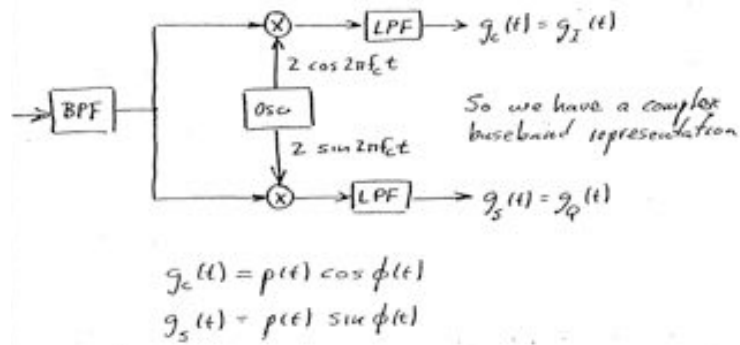
$$g_s(t) = p(t) \sin \phi(t)$$

Now consider the complex envelope function

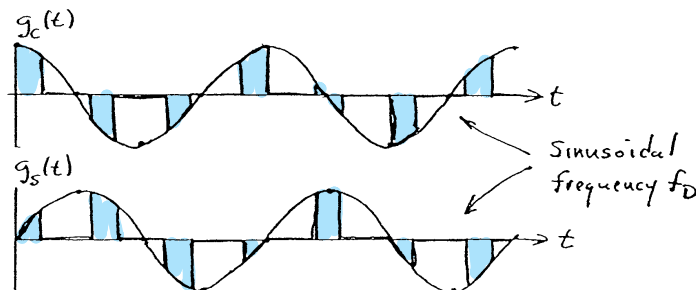
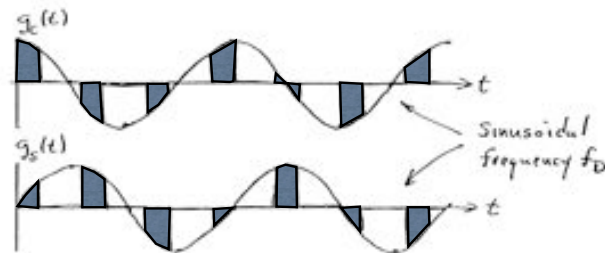
$$u(t) = g_c(t) + i g_s(t) = p(t) \cos \phi(t) + i p(t) \sin \phi(t)$$

$$= p(t) [\cos \phi(t) + i \sin \phi(t)]$$

$$= p(t) e^{i \phi(t)}. \quad (\text{By Euler's identity})$$



For $\phi(t) = 2\pi f_D t$ as in our case, the two signals appear as follows:

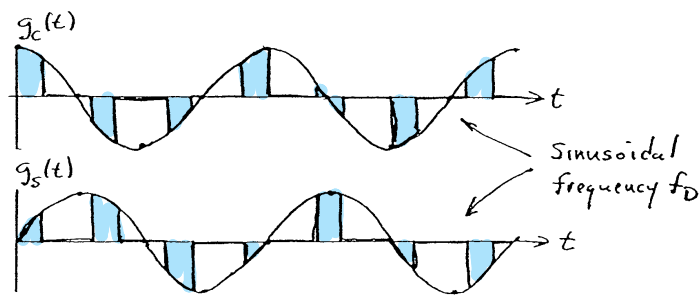


Now we can form the complex envelope function

$$u(t) = g_c(t) + i g_s(t)$$

If we sample $u(t)$ at sampling instants $t_0, t_1, \dots, t_{N-1}$ such that $t_n - t_{n-1} = T$ with t_0 selected within the first pulse ($0 < t_0 < t_p$) we get one complex sample per transmitted pulse

$$\begin{aligned} u(t_n) &= g_c(t_n) + i g_s(t_n) \\ &= p(t_n) \exp \{i \phi(t_n)\} \\ &= p(t_n) \exp \{i 2\pi f_D t_n\} \\ &= \exp \{i 2\pi f_D t_n\} \end{aligned} \quad (\text{single target case})$$

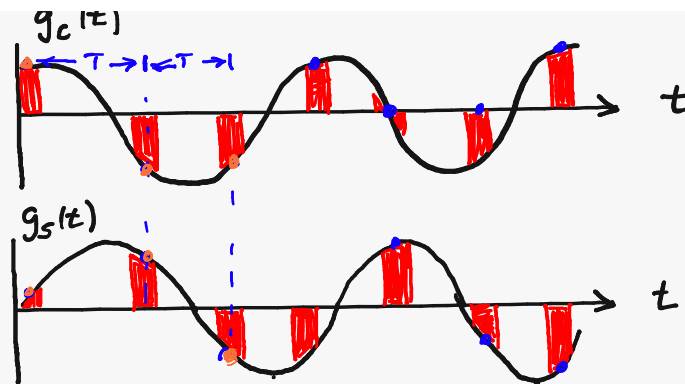


Now we can form the complex envelope function

$$u(t) = g_c(t) + i g_s(t)$$

If we sample $u(t)$ at sampling instants $t_0, t_1, \dots, t_{N-1}$ such that $t_n - t_{n-1} = T$ with t_0 selected within the first pulse ($0 < t_0 < t_p$) we get one complex sample per transmitted pulse

$$\begin{aligned} u(t_n) &= g_c(t_n) + i g_s(t_n) \\ &= p(t_n) \exp \{ i \phi(t_n) \} \\ &= p(t_n) \exp \{ i 2\pi f_0 t_n \} \\ &= \exp \{ i 2\pi f_0 t_n \} \quad (\text{single target case}) \end{aligned}$$



Now we can form the complex envelope function

$$u(t) = g_c(t) + i g_s(t)$$

If we sample $u(t)$ at sampling instants $t_0, t_1, \dots, t_{N-1}$ such that

$$t_n - t_{n-1} = T, \quad n = 1, \dots, N-1$$

with t_0 selected within the first pulse ($0 < t_0 < t_p$) we get one complex sample from each pulse return

16.14

Note that if we define the complex sequence

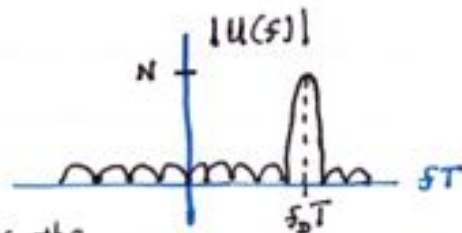
$$u_n = u(t_n) = u(t_0 + nT), \quad n = 0, 1, \dots, N-1$$

and compute the DTFT of $\{u_n\}$, we get

$$\begin{aligned} U(f) &= \sum_{n=0}^{N-1} u_n e^{-i2\pi nTf} = \sum_{n=0}^{N-1} e^{i2\pi f_0(t_0 + nT)} e^{-i2\pi nTf} \\ &= e^{i2\pi f_0 t_0} \sum_{n=0}^{N-1} e^{-i2\pi(f - f_0)nT} \end{aligned}$$

$$\text{For } f = f_0 \Rightarrow |U(f)| = N,$$

$$\text{For } f \neq f_0 \Rightarrow |U(f)| \approx 0.$$



n.b where we pick t_0 determines the delay to the target. (t_0 = delay to first pulse return.)

16.15

We could calculate the DFT of the sequence u_n :

$$U_k = \sum_{n=0}^{N-1} u_n e^{-i\frac{2\pi nk}{N}}, \quad k = 0, 1, \dots, N-1.$$

Here again, we would get a large magnitude when $\frac{k}{N} \approx Tf_0$:

$$U_k = \begin{cases} N \cdot e^{i2\pi f_0 t_0}, & k \approx NTf_0 \\ 0, & \text{elsewhere.} \end{cases}$$

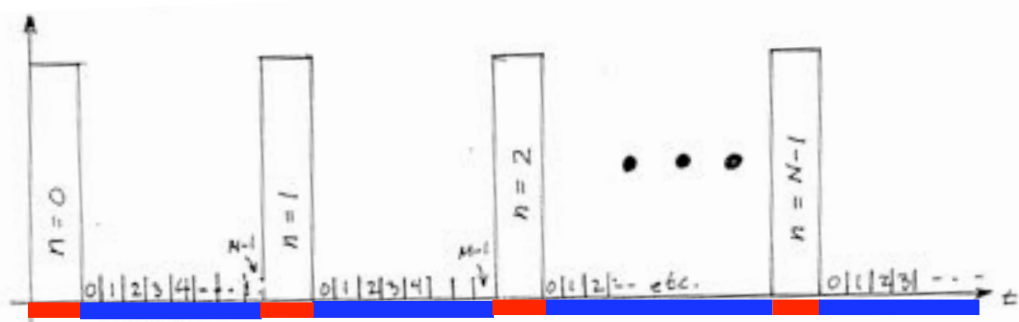
So for ease of implementation, the DFT (Computed using an FFT) is a common approach to determining the Doppler signature of a target located at a range corresponding to sampling time t_0 .

This is the beginning (and main idea) on which we build the delay-Doppler processor, for a pulse train.

The Pulse Waveform Delay-Doppler Processor

In general, we are interested in looking at the returns from all possible ranges (delays).

The situation appears as follows:



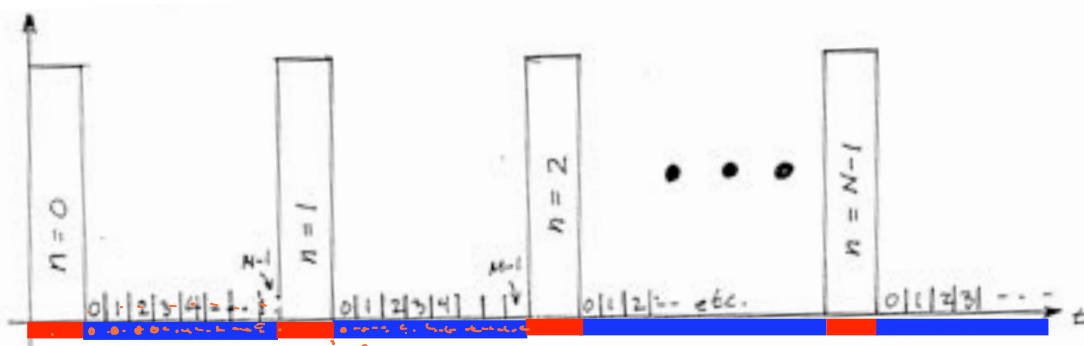
Because we transmit finite-bandwidth signals, we only have limited range or delay resolution.

Recall what $\beta_s(\tau, \nu)$ looks like.

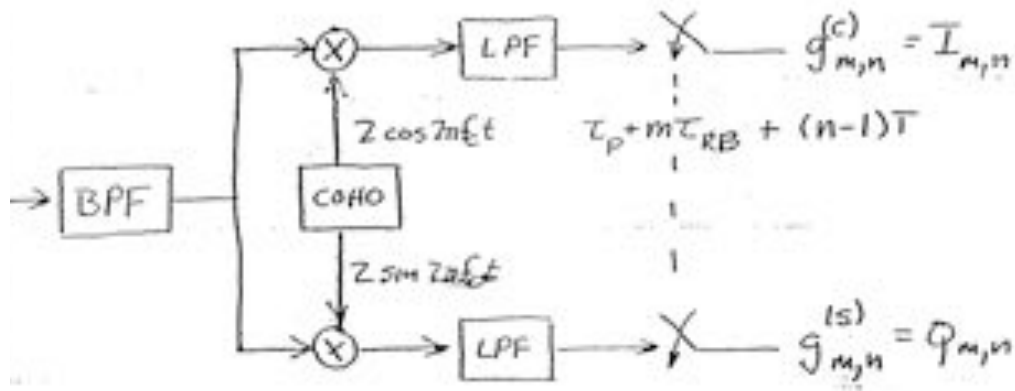
So we will divide the listening period into bins of duration τ_{RB} .

So we will divide the listening period into time bins of width τ_{RB} and sample the signal once per bin for each range bin of interest.

Label the range bins $0, 1, \dots, M-1$.



We can get the sample returns from each of the range bins (both I and Q components to represent the complex baseband signal) by processing the received signal as follows:



n.b., τ_p is the transmitted pulse duration.

Once we have carried out this procedure, we can think of the I and Q samples $\{I_{m,n}, Q_{m,n}\}$ as being arranged in an array as follows:

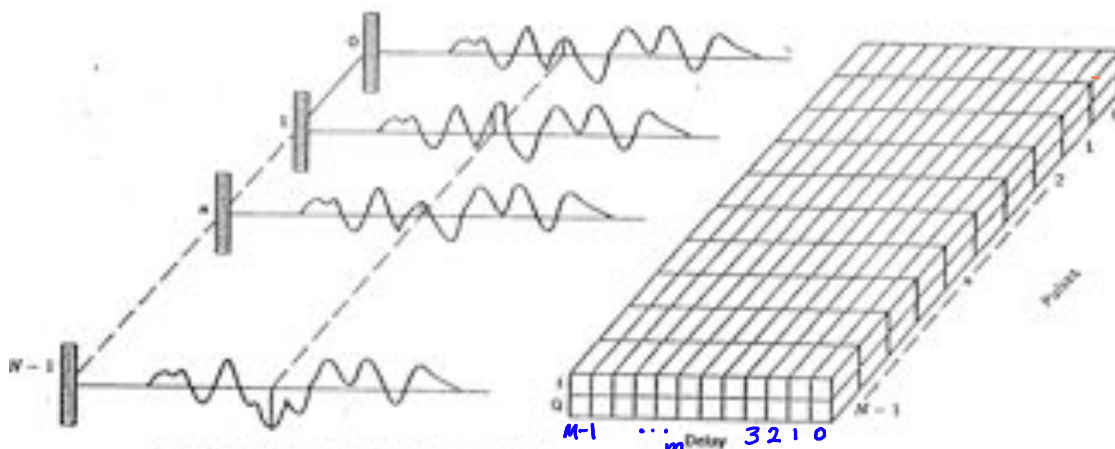
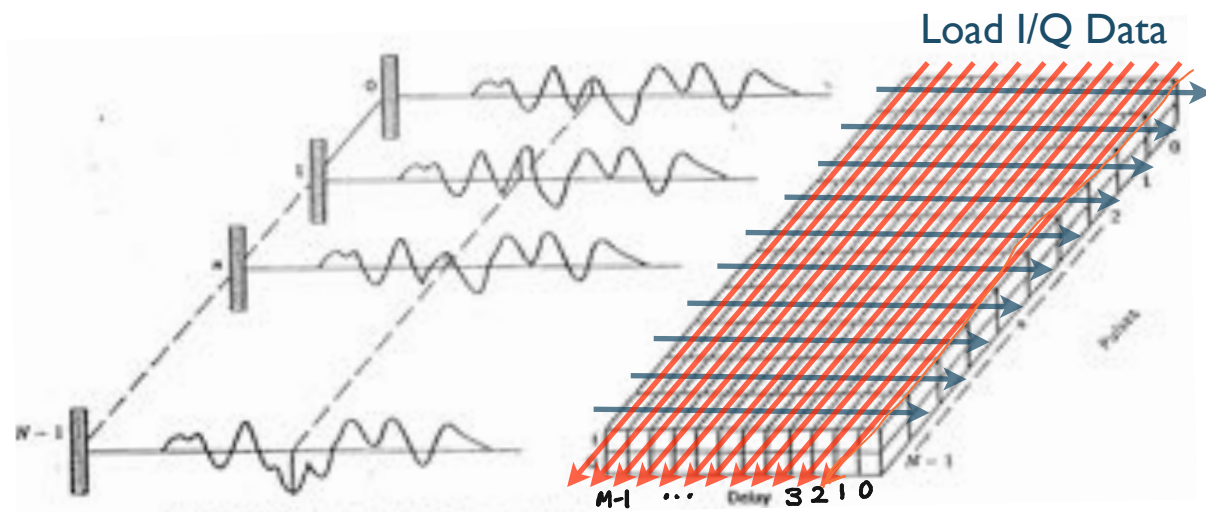


Figure 10.3 A conceptual diagram of the accumulation of I & Q samples at various delays, from N consecutive pulses.

(From: N. Levanon, *Radar Principles*, Wiley, 1987)



Compute DFT for each
range cell

So we can collect the I and Q samples into the arrays $[I_{mn}]$ and $[Q_{mn}]$, resp., taking samples at times

$$t_{m,n} = t_p + mT_{RB} + nT$$

n.b $T_{RB} \ll T$

$$\left. \begin{array}{l} T_{RB} \sim 1\mu S \\ T \sim 100\mu S \end{array} \right\} \text{Typical!}$$

For a given range cell (fixed m), once we have the N complex samples

$$I_{mn} + iQ_{mn}, \quad n=0, 1, \dots, N-1$$

we can carry out Doppler processing using the DFT.

Now suppose we have a signal $g(t)$. Its Fourier transform is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-i2\pi f t} dt.$$

Suppose $g(t)$ is $[-W, W]$ -BL. $\Rightarrow G(f) = 0, \forall f \notin [-W, W]$

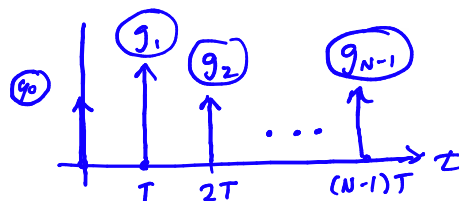
Assume we sample $g(t)$ at times

$$t_n = nT, \quad n = 0, \dots, N-1; \quad \text{where } T = \frac{1}{2W}$$

$$\Rightarrow g_n = g(t_n) = g(nT), \quad n = 0, \dots, N-1.$$

The sampled version of the waveform can be represented by the

$$\tilde{g}(t) = \sum_{n=0}^{N-1} g_n S(t - nT).$$

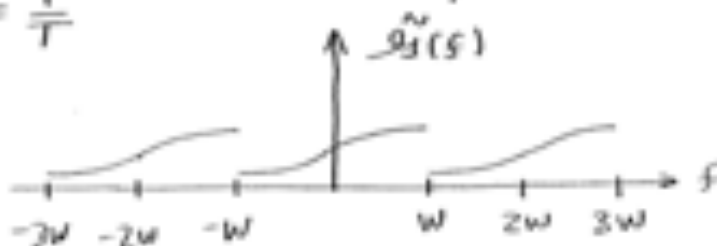


The Fourier transform of $\tilde{g}(t)$ is

$$\begin{aligned} \tilde{g}_f(f) &= \int_{\mathbb{R}} \left[\sum_{n=0}^{N-1} g_n S(t - nT) \right] e^{-i2\pi f t} dt \\ &= \sum_{n=0}^{N-1} g_n \int_{-\infty}^{\infty} S(t - nT) e^{-i2\pi f t} dt = \sum_{n=0}^{N-1} g_n e^{-i2\pi f nT} \end{aligned}$$

i.e. $\tilde{g}_f(f)$ is the DTFT of $\{g_n\}_{n=0}^{N-1}$ DTFT of $\{g_0, g_1, \dots, g_{N-1}\}$

We note that $\tilde{g}_f(f)$ is periodic with period $2W = \frac{1}{T}$



We can span the whole frequency range $[-W, W]$ or equivalently $[0, 2W]$ by sampling N different frequency points

$$f_k = k \cdot \frac{2W}{N} = \frac{k}{NT}, \quad k = 0, 1, \dots, N-1.$$

Thus we have that

$$\tilde{g}_s(f_k) = G_k = \sum_{n=0}^{N-1} g_n e^{-i2\pi \left(\frac{k}{NT}\right) nT} = \sum_{n=0}^{N-1} g_n e^{-i2\pi \frac{kn}{N}}$$

i.e. This is just the DFT of $\{g_n\}_{n=0}^{N-1}$,

$$\text{Since } \text{DFT}_N^{-1}(\{G_k\}) = \{g_n\}$$

$$\text{DFT}_N(\{g_n\}) = G_k$$

$$\text{where } G_k = \sum_{n=0}^{N-1} g_n e^{-i2\pi kn/N} \text{ and } g_n = \frac{1}{N} \sum_{k=0}^{N-1} G_k e^{i2\pi kn/N}.$$

for $k = 0, 1, 2, \dots, N-1$

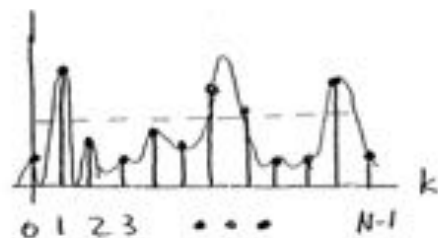
for $n = 0, 1, \dots, N-1$

As we saw before in the DTFT, if a signal $g(t)$ ¹¹⁰ has Doppler shift f_D that is near $\frac{k}{NT}$ for some integer k , then

$G_k = \tilde{g}_s(f_k)$ will have a large response.

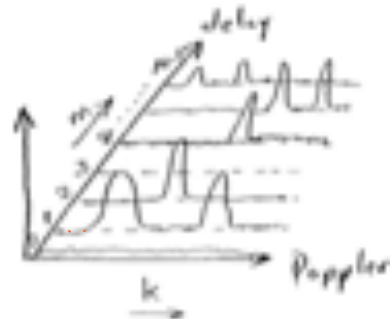
→ One proportional to $\sqrt{\sigma^2}$.

If we have a number of different targets with Doppler frequencies $f_{D_1}, \dots, f_{D_M}$, we will have a peak in the Doppler spectrum corresponding to each one.



16.28

If we process in this way for each range cell (each m), we can construct a "Delay-Doppler" spectrum.



From such a Delay-Doppler image one can detect where the targets are in Delay and Doppler (E and D).