Session 13

Recall...

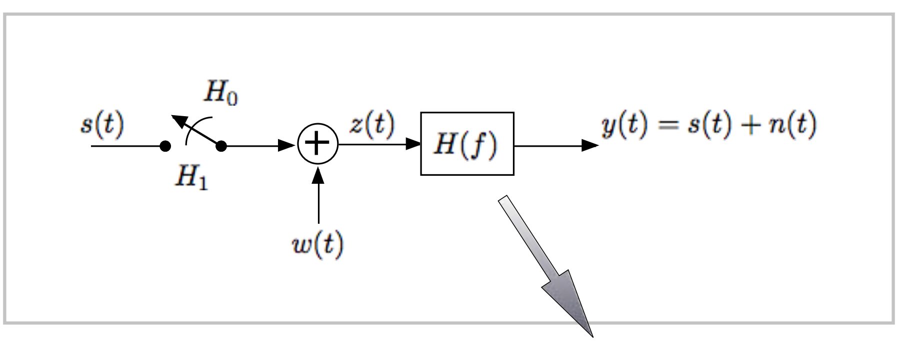
13.1

Detection of a Known Signal in Additive White Gaussian Noise

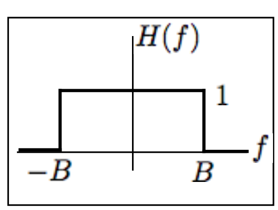
Suppose we have a signal $s(t)$ of known duration T in the interval $[0, T]$ such that

$$s(t) = 0, \quad t \notin [0, T].$$

We wish to determine whether or not this signal is present in the presence of *Additive White Gaussian Noise* (AWGN).



Hypothetical Lowpass Filter:



Recall... 13.2

Assume the noise $w(t)$ is zero-mean Gaussian white noise having (two-sided) PSD

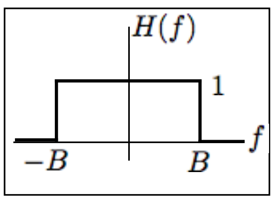
$$S_{ww}(f) = \frac{N_0}{2}. \qquad \left(= \frac{kT_e}{2} \right)$$

We want to determine which of two possible hypotheses are in effect:

$$\begin{aligned} H_0 &: r(t) = w(t) && \text{(target absent)}, \\ H_1 &: r(t) = s(t) + w(t) && \text{(target present)}. \end{aligned}$$

Assume we observe the process $z(t)$ through a hypothetical lowpass filter

$$H(f) = 1_{[-B,B]}(f) = \begin{cases} 1, & \text{for } |f| \leq B; \\ 0, & \text{for } |f| > B. \end{cases}.$$



Assume B sufficiently large such that all but a negligible fraction of the energy in $s(t)$ passes through $H(f)$.

Recall... 13.3

If we input only the white noise $w(t)$ into filter $H(f)$, the output $n(t)$ becomes bandlimited white noise $n(t)$:

$$\mathrm{E}[n(t)] = 0$$

$$S_{nn}(f) = \frac{N_0}{2} \cdot 1_{[-B,B]}(f)$$

$$\begin{aligned} R_{nn}(\tau) &= N_0 B \left(\frac{\sin 2\pi B\tau}{2\pi B\tau} \right) \\ &= N_0 B \operatorname{sinc}(2B\tau). \end{aligned}$$

It follows that

$$R_{nn}(\tau) = 0, \quad \text{for} \quad \tau = \frac{\pm 1}{2B}, \frac{\pm 2}{2B}, \frac{\pm 3}{2B}, \ldots$$

$$\Longrightarrow \mathrm{E}\left[n\left(t_0 + \frac{k}{2B}\right) \cdot n\left(t_0 + \frac{m}{2B}\right) \right] = N_0 B \delta_{k,m} = \begin{cases} N_0 B, & \text{for } k = m, \\ 0, & \text{for } k \neq m. \end{cases}$$

$$\forall t_0 \in \mathbf{R}$$

Recall... 13.4

Thus samples of the random process $n(t)$ taken at increments of $\Delta t = 1/(2B)$ form a sequence of uncorrelated Gaussian random variables.

Because this sequence is both Gaussian and uncorrelated, it follows that it is a sequence of independent Gaussian random variables.

Thus $n(t_0 + 1/(2B)), n(t_0 + 2/(2B)), \ldots, n(t_0 + M/(2B))$ are i.i.d. Gaussian random variables with mean zero and variance $\sigma_n^2 = R_{nn}(0) = N_0 B$.

If we take $t_0 = 0$ and sample at time instants $t_m = m/(2B)$, where $m = 1, 2 \ldots 2BT$ over duration T, the p.d.f. under H_0 is

$$\begin{aligned} f_0(y(t_1), \ldots, y(t_{2BT})) &= \prod_{m=1}^{2BT} \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left\{-\frac{y^2(t_m)}{2\sigma_n^2}\right\} \\ &= \frac{1}{(2\pi)^{BT}\sigma_n^{2BT}} \exp\left\{-\frac{1}{2\sigma_n^2}\sum_{m=1}^{2BT} y^2(t_m)\right\}. \end{aligned}$$

Recall... 13.5

The p.d.f. under H_1 is

$$\begin{aligned} f_1(y(t_1), \ldots, y(t_{2BT})) &= \prod_{m=1}^{2BT} \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left\{-\frac{(y(t_m) - s(t_m))^2}{2\sigma_n^2}\right\} \\ &= \frac{1}{(2\pi)^{BT}\sigma_n^{2BT}} \exp\left\{-\frac{1}{2\sigma_n^2}\sum_{m=1}^{2BT} (y(t_m) - s(t_m))^2\right\}. \end{aligned}$$

The log-likelihood ratio as

$$\begin{aligned} \ell(\underline{Y}) &= \log\left(\frac{f_1(y(t_1)\ldots y(t_{2BT}))}{f_0(y(t_1)\ldots y(t_{2BT}))}\right) \\ &= \log\left(\frac{\exp\{-\frac{1}{2\sigma_n^2}\sum_{m=1}^{2BT}(y(t_m) - s(t_m))^2\}}{\exp\{-\frac{1}{2\sigma_n^2}\sum_{m=1}^{2BT} y^2(t_m)\}}\right) \\ &= -\frac{1}{2\sigma_n^2}\left[\sum_{m=1}^{2BT}(y(t_m) - s(t_m))^2 - \sum_{m=1}^{2BT} y^2(t_m)\right]. \end{aligned}$$

Recall...

Thus the most powerful test of size $\alpha = P_{\text{FA}}$ is of the form

$$\frac{1}{2\sigma_w^2} \sum_{m=1}^{2BT} \left[2y(t_m)s(t_m) - s^2(t_m)\right] \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_0,$$

where γ_0 is a threshold determined by the required false alarm rate P_{FA}.

Equivalently, we can write the test as

$$\frac{1}{N_0 B} \sum_{m=1}^{2BT} y(t_m)s(t_m) \underset{H_0}{\overset{H_1}{\gtrless}} \log \gamma_0 + \frac{1}{2N_0 B} \sum_{m=1}^{2BT} s^2(t_m).$$

Recall...

If we let $\Delta t = 1/(2B)$, we can rewrite

$$\frac{1}{N_0 B} \sum_{m=1}^{2BT} y(t_m)s(t_m) \underset{H_0}{\overset{H_1}{\gtrless}} \log \gamma_0 + \frac{1}{2N_0 B} \sum_{m=1}^{2BT} s^2(t_m)$$

as

$$\frac{2\Delta t}{N_0} \sum_{m=1}^{2BT} y(t_m)s(t_m) \underset{H_0}{\overset{H_1}{\gtrless}} \log \gamma_0 + \frac{\Delta t}{N_0} \sum_{m=1}^{2BT} s^2(t_m).$$

Now if we let the bandwidth B of the ideal low-pass filter grow arbitrarily large, i.e., $B \to \infty$, then $\Delta t = 1/(2B) \to 0$, and this Riemann sum can be replaced by the Riemann integral

$$\frac{2}{N_0} \int_0^T y(t)s(t)\,dt \underset{H_0}{\overset{H_1}{\gtrless}} \log \gamma_0 + \frac{1}{N_0} \int_0^T s^2(t)\,dt,$$

or equivalently,

$$\frac{2}{N_0} \int_0^T y(t)s(t)\,dt \underset{H_0}{\overset{H_1}{\gtrless}} \log \gamma_0 + \frac{E_s}{N_0}.$$

n.b., $E_s = \int_0^T s^2(t)\,dt$.

Recall...

This now shows us how to implement a system to compute the optimal decision statistic for our detection problem. Let

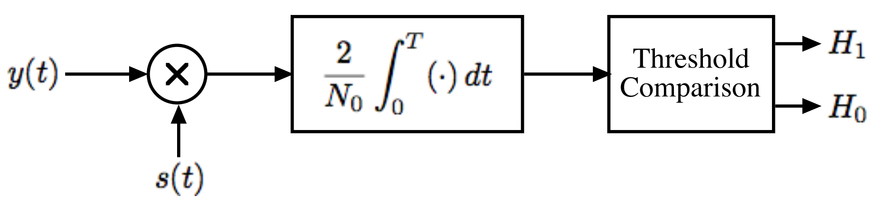
$$x(t) = \frac{2}{N_0} \int_0^t y(\tau)s(\tau)\, d\tau.$$

Then the most powerful test having a given P_{FA} compares the random variable $x(T)$ to a threshold $\gamma_1 = \log \gamma_0 + E_s/N_0$.

The random process $x(t)$ is a scaled version of the time cross-correlation between the transmitted signal $s(t)$ and $y(t) \approx r(t)$ as $B \to \infty$.

Recall...

So the processor and optimal test can be implemented using a correlator:



Note that this correlator implementation requires the synchronization of the reference signal $s(t)$ with the signal component in the incoming signal $y(t)$.

In a digital signal processing (DSP) implementation, this can done with a resynchronized version of $s(t)$ repeated in each sampling interval.

While this works, it is not the most computationally effiecient approach.

However, we can eliminate the synchronization issue altogetherby implementing the correlator using a linear time-invariant (LTI) filter.

We can implement the correlation using the linear time-invariant (LTI) filter

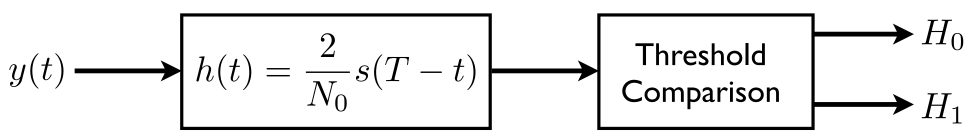
$$h(t) = \frac{2}{N_0} s(T - t)$$

to replace the correlator, because

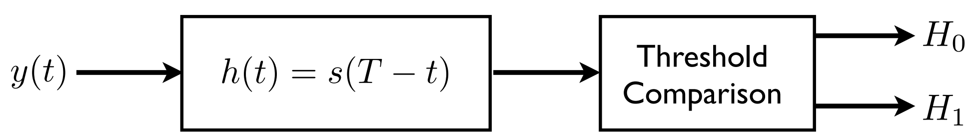
$$\begin{aligned} x(T) &= h(t) * y(t)|_{t=T} \\ &= \int_0^T h(\tau) y(T-\tau)\, d\tau \\ &= \int_0^T \frac{2}{N_0} s(T-\tau) y(T-\tau)\, d\tau \\ &= \boxed{\frac{2}{N_0} \int_0^T s(\tau) y(\tau)\, d\tau.} \end{aligned}$$

Correlator output at time *T*

In block diagram form, the LTI filter implementation appears as follows:



or eliminating the scale factor $2/N_0$ and adjusting the thresho9ld accordingly:



This implementation is called the *matched filter* implementation.

Because $h(t)$ is a LTI filter, the synchronization issue is not as critical here.

Computing the Threshold

In order to compute the threshold for the test that uses the statistic $x(T)$, we need to compute the mean and variance of the Gaussian random variable $x(T)$ under each hypothesis:

$$H_0 : \mathrm{E}[x(T)] = \frac{2}{N_0} \int_0^T \mathrm{E}[n(t)s(t)]\,dt = 0,$$

$$H_1 : \mathrm{E}[x(T)] = \frac{2}{N_0} \int_0^T E[(s(t) + n(t))s(t)]\,dt = \frac{2}{N_0} \int_0^T |s(t)|^2 dt = \frac{2E_s}{N_0}.$$

The variance is the same under both hypotheses, and is given by

$$\begin{aligned} \sigma^2_{x(T)} &= \mathrm{E}_{H_0}[x(T)^2] \\ &= \mathrm{E}\left[\frac{2}{N_0}\int_0^T n(u)s(u)\,du \cdot \frac{2}{N_0}\int_0^T n(v)s(v)\,dv\right] \\ &= \frac{4}{N_0^2}\int_0^T\int_0^T E[n(u)n(v)]s(u)s(v)\,du\,dv. \end{aligned}$$

Because $n(t)$ is bandlimited white noise over frequency band $[-B, B]$, as $B \to \infty$,

$$\mathrm{E}[n(u)n(v)] = R_{nn}(u - v) = \frac{N_0}{2}\delta(u - v).$$

Substituting this result into

$$\sigma^2_{x(T)} = \frac{4}{N_0^2}\int_0^T\int_0^T E[n(u)n(v)]s(u)s(v)\,du\,dv,$$

the variance is given by

$$\begin{aligned} \sigma^2_{x(T)} &= \frac{4}{N_0^2}\int_0^T\int_0^T \frac{N_0}{2}\delta(u - v)s(u)s(v)\,du\,dv \\ &= \frac{N_0}{2}\int_0^T s^2(u)\,du \\ &= \frac{2E_s}{N_0}. \end{aligned}$$

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So we have

$$H_0 : x(T) \sim \mathcal{N}\left(0, \frac{2E_s}{N_0}\right),$$
$$H_1 : x(T) \sim \mathcal{N}\left(\frac{2E_s}{N_0}, \frac{2E_s}{N_0}\right).$$

It follows the most powerful test of size α is

$$x(T) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_1,$$

with the threshold γ_1 computed by noting that

$$\begin{aligned} \alpha &= P_{FA} \\ &= \int_{\gamma_1}^{\infty} \frac{1}{\sqrt{4\pi E_s/N_0}} \exp\left\{-\frac{u^2}{4E_s/N_0}\right\} du \\ &= 1 - \Phi\left(\frac{\gamma_1}{\sqrt{2E_s/N_0}}\right), \end{aligned}$$

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The threshold can then be found as

$$\gamma_1 = \sqrt{\frac{2E_s}{N_0}} \Phi^{-1}(1-\alpha),$$

where

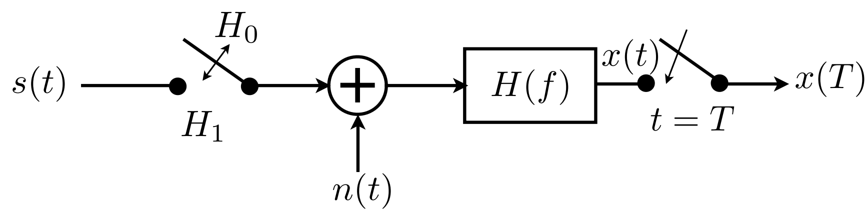
$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du.$$

The power of the test is

$$\begin{aligned} \beta &= P_D \\ &= \int_{\gamma_1}^{\infty} \frac{1}{\sqrt{4\pi E_s/N_0}} \exp\{-\frac{(u - E_s/N_0)^2}{4E_s/N_0}\} du \\ &= 1 - \Phi\left(\frac{\gamma_1 - 2E_s/N_0}{\sqrt{2E_s/N_0}}\right) \\ &= 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \sqrt{2E_s/N_0}\right). \end{aligned}$$

The Matched Filter: Signal-to-Noise Ratio Maximization

An alternative approach to deriving—and extending—the matched filter is to consider the signal-to-noise ratio at the LTI filter output at time $t = T$:



Assume $s(t)$ has duration T:

$$s(t) = 0, \qquad t \notin [0, T].$$

Assume $s(t)$ has energy E:

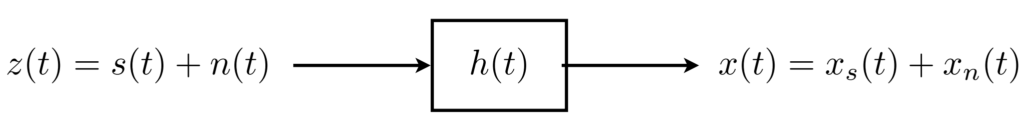
$$\int_0^T |s(t)|^2 \, dt = E.$$

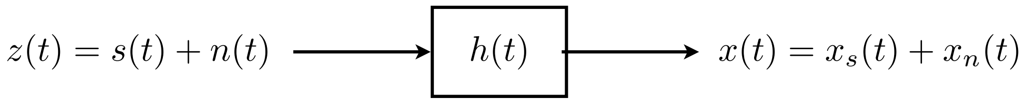
We will assume that the noise $n(t)$ is a zero-mean wide-sense stationary (WSS) random process with power spectral density $S_{nn}(f)$.

- $n(t)$ is not necessarily Gaussian.
- $n(t)$ is not necessarily white.

We wish to find the LTI filter $h(t) \Leftrightarrow H(f)$ that maximizes the signal-to-noise ratio (SNR) at its output at time $t = T$.

Question: What is the SNR at the filter output at time $t = T$?





Computing the filter output $x(T)$ at time $t = T$, we have

$$\begin{aligned}
x(T) &= \int_{-\infty}^{\infty} h(\tau) z(T-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\left[s(T-\tau) + n(T-\tau)\right] d\tau \\
&= \int_{-\infty}^{\infty} h(\tau) s(T-\tau)\, d\tau + \int_{-\infty}^{\infty} h(\tau) n(T-\tau)\, d\tau \\
&= x_s(T) + x_n(T)
\end{aligned}$$

The SNR of the filter output $x(T)$ at time $t = T$ is defined as

$$\mathrm{SNR} \equiv \frac{|x_s(T)|^2}{\mathrm{E}[|x_n(T)|^2]}.$$

Our problem becomes finding the filter $h(t)$ that maximizes the SNR at time T:

$$h_M(t) = \arg\max_{h(t)} \left[\frac{|x_s(T)|^2}{\mathrm{E}[|x_n(T)|^2]}\right].$$

By Parseval's theorem,

$$|x_s(T)|^2 = \left|\int_{-\infty}^{\infty} h(\tau) s(T-\tau)\right|^2 = \left|\int_{-\infty}^{\infty} H(f) S(f) e^{i2\pi T f}\, df\right|^2,$$

and

$$\mathrm{E}[|x_n(T)|^2] = R_{x_n x_n}(0) = \int_{-\infty}^{\infty} S_{x_n x_n}(f)\, df = \int_{-\infty}^{\infty} |H(f)|^2 S_{nn}(f)\, df.$$

Thus

$$\begin{aligned}
\mathrm{SNR} &= \frac{\left|\int_{-\infty}^{\infty} H(f) S(f) e^{i2\pi T f} df\right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 S_{nn}(f)\, df} \\
&= \frac{\left|\int_{-\infty}^{\infty} H(f)\sqrt{S_{nn}(f)}\, \frac{S(f)}{\sqrt{S_{nn}(f)}} e^{i2\pi T f} df\right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 S_{nn}(f)\, df}.
\end{aligned}$$

By the Schwarz inequality, we know that any two square-integrable functions f and g satisfy

$$\left|\int f(t)g(t)\,dt\right|^2 \le \int |f(t)|^2\,dt \cdot \int |g(t)|^2\,dt,$$

with equality if and only if $f(t) = kg^*(t)$.

Thus we have

$$\begin{aligned} \text{SNR} &= \frac{\left|\int_{-\infty}^{\infty} H(f)\sqrt{S_{nn}(f)}\frac{S(f)}{\sqrt{S_{nn}(f)}}e^{i2\pi Tf}\,df\right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 S_{nn}(f)\,df} \\ &\le \frac{\int_{-\infty}^{\infty} |H(f)|^2 S_{nn}(f)\,df \cdot \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_{nn}(f)}\,df}{\int_{-\infty}^{\infty} |H(f)|^2 S_{nn}(f)\,df} \\ &= \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_{nn}(f)}\,df, \end{aligned}$$

Apply Schwarz Inequality to numerator

with equality if and only if

$$\boxed{H(f) = \frac{kS^*(f)}{S_{nn}(f)}e^{-i2\pi Tf},}$$

Matched Filter for wide-sense stationary additive colored noise

where k is a complex constant.

$$\boxed{H(f) = \frac{kS^*(f)}{S_{nn}(f)}e^{-i2\pi Tf}.}$$

Note that when the noise is white, $S_{nn}(f) = N_0/2$, and this becomes

$$H(f) = \frac{2k}{N_0}S^*(f)e^{-i2\pi fT}.$$

The corresponding impulse response of is

$$h(t) = \frac{2k}{N_0}s(T-t),$$

and the resulting SNR is

$$\text{SNR} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{N_0/2}\,df = \frac{2E_s}{N_0}.$$

n.b., the exact same result we got for additive white Gaussian noise.

It can be shown that at threshold test on the output of this filter,

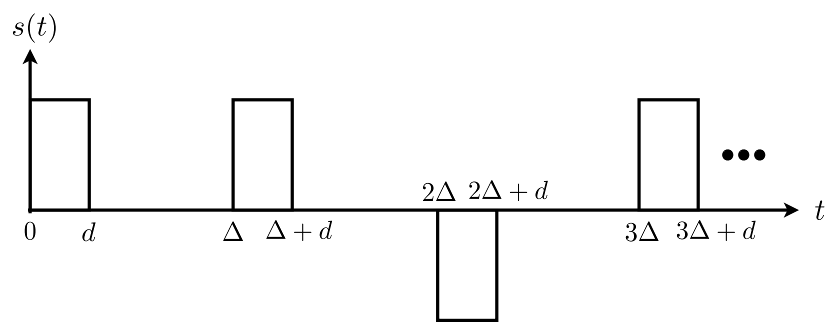
$$H(f) = \frac{kS^*(f)}{S_{nn}(f)} e^{-i2\pi Tf},$$

yields the most powerful test of size α in wide-sense stationary Gaussian noise with power PSD $S_{nn}(f)$.

This is because

- Under hypothesis H_0, the output $x(T)$ at time $t = T$ is a Gaussian random variable with mean 0.
- Under hypothesis H_1, the output $x(T)$ at time $t = T$ is a Gaussian random variable with mean proportinal to the signal-to-noise ratio SNR.
- The variance in the output $x(T)$ is identical under both hypotheses.
- In such a Gaussian detection problem, ther performace improves as the difference of the means becomes larger.
- In this situation, the difference of the means is maximized when the SNR is maximized.
- The SNR is maximized by the above filter.

Detection of a Pulse Train in Stationary Noise

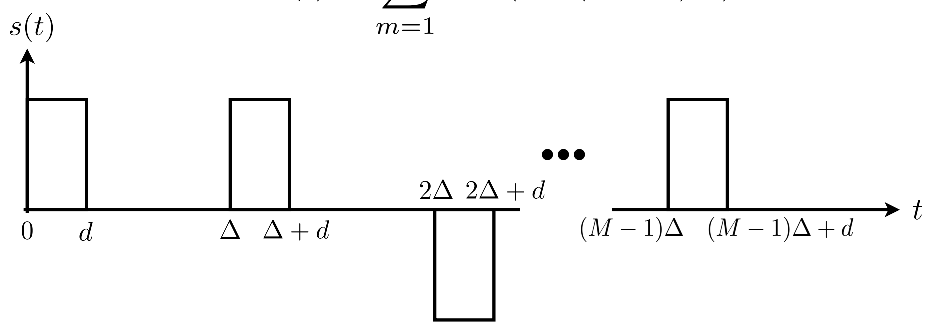


- Suppose we have a waveform $s(t)$ constructed by periodically repeating some basic pulse shape $p(t)$.
- Assume we may amplitude modulate the individual pulses.
- Assume the pulses are non-overlapping.

13.23

Consider the waveform $s(t)$ constructed by regularly repeating a basic wavefor $p(t)$ with amplitude modulation on each pulse:

$$s(t) = \sum_{m=1}^{M} a_m p(t - (m-1)\Delta).$$



- Here a_m is the amplitude of the m-th pulse.
- The amplitudes could be *binary*(*e.g.*,± 1), *real* or *complex*.
- Assume we receive the signal in the presence of zero-mean stationary noise having PSD $S_{nn}(f)$.
- We want to design the *matched filter* to maximze the output SNR at observation time $t = T$.

13.24

We know that in this situation, if

$$S(f) = \mathcal{F}\{s(t)\} = \int_{-\infty}^{\infty} s(t) e^{-i2\pi ft}\, dt,$$

Then the matched filter is given by

$$\tilde{H}_M(f) = \frac{S^*(f) e^{-i2\pi fT}}{S_{nn}(f)}.$$

If we take $T = M\Delta$ so that the observation time trailing the last pulse is equal to the observation time between all other pulses, and we note that

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-i2\pi ft}\, dt = \sum_{m=1}^{M} a_m P(f) e^{-i2\pi f(m-1)\Delta},$$

where

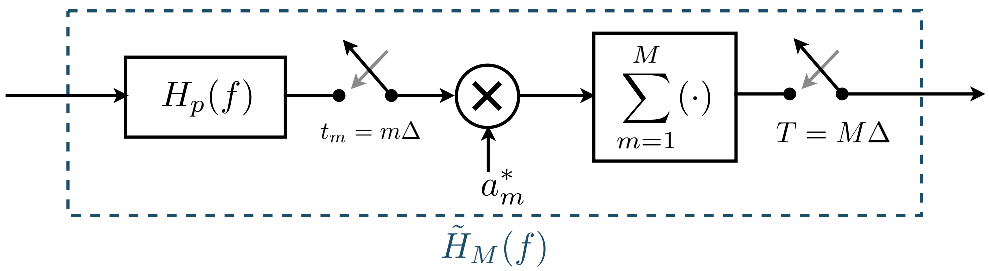
$$P(f) = \int_{-\infty}^{\infty} p(t) e^{-i2\pi ft}\, dt.$$

So the matched filter can be written as

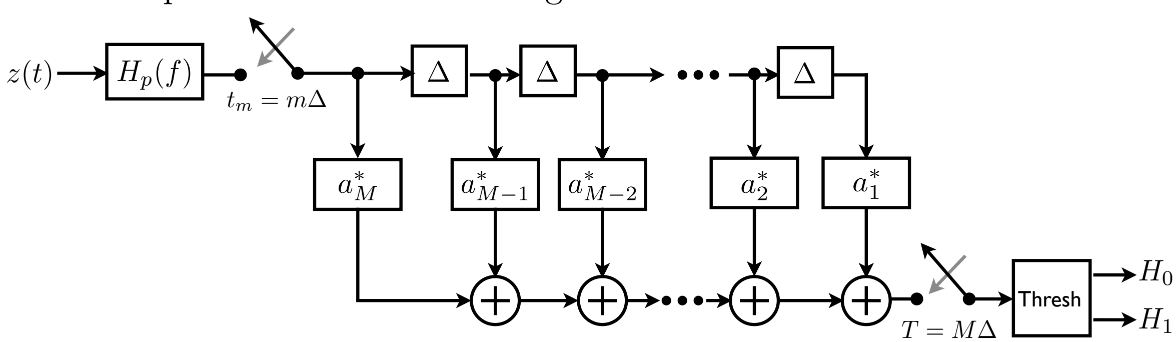
$$
\begin{aligned}
\tilde{H}_M(f) &= \frac{S^*(f)}{S_{nn}(f)} e^{-i2\pi f M\Delta} \\
&= \sum_{m=1}^{M} a_m^* \frac{P^*(f)}{S_{nn}(f)} e^{-i2\pi f \Delta} e^{-i2\pi (M-m) f \Delta} \\
&= \sum_{m=1}^{M} a_m^* H_p(f) e^{-i2\pi (M-m) f \Delta},
\end{aligned}
$$

where $H_p(f)$ is the single pulse matched filter sampled at time $t = \Delta$ and given by

$$H_p(f) = \frac{P^*(f)}{S_{nn}(f)} e^{-i2\pi f \Delta}.$$



We can implement this in block diagram form as follows:



Or it can be implemented "all-analog" using tapped delay lines: