Contiguous Pulse Binary Integration Analysis

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The probability of detecting m or more pulses contiguously—that is, in a row—from a pulse train of n pulses is determined when the detection of each pulse is an independent Bernoulli trial with probability p. While a general closed-form expression for this probability is not known, we present an analytical procedure that gives the exact expression for the probability of interest for any particular case. We also present simple asymptotic expressions for these probabilities and develop bounds on the probability that the number of pulses that must be observed before m contiguous detections is greater than or less than some particular number. We consider the implications for binary integration in radar and electronic warfare problems.

Manuscript received June 29, 1993; revised June 6, 1994.

IEEE Log No. T-AES/32/3/05865.

This work was supported by NSF Research Initiation Award MIP-9010834.

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I. INTRODUCTION

The problem of detecting the presence of a number of successive or contiguous pulses in a pulse train is of general interest in a number of radar and electronic warfare problems. Levanon [1, pp. 58–60] gives an example of a radar binary integration scheme that is a special case of this problem. He considers the calculation of the probability of detection of two or more pulses in a row out of four pulses, where the detection of each pulse is considered to be an independent Bernoulli trial with probability p of success. He does not, however, consider the generalization of detecting m or more pulses in a row from a pulse train of n pulses. Here we consider this more general problem.

In contrast to requiring the contiguous detection of m out of n pulses, there is the common binary integration technique of requiring u out of n pulses irrespective of order. The cumulative probability of detection (and false alarm) in this case is easy to calculate in terms of the binomial distribution. Hence we refer to this scheme as the *binomial binary integration*. In the situation where the individual pulse detections are statistically independent, one would not expect the cumulative pulse detection scheme to be superior to a simple m out of n decision rule. This is because if individual detections are Bernoulli trials with probability p of success, it is well known that the ratio of the number of successes to the total number of trials is a sufficient statistic for (as well as the *minimum variance unbiased estimator* of) the probability of success on any single trial [2, 3]. Nevertheless, the comparative performance of the contiguous detection scheme is of interest. This is because there are some radar target detection problems in which contiguous pulse detection is of interest. For example, if measurements are being made with a high-resolution (in range, Doppler, azimuth, or elevation) radar system in which a typical target of interest spans several contiguous resolution cells, a binary integration scheme requiring detections in contiguous resolution cells is appropriate. Although in such a situation detections in individual resolution cells may not be well modeled as independent Bernoulli trials, it is of interest to see the degradation in performance of the contiguous resolution cell scheme over that of binomial binary integration.

Another area where the problem of detecting m or more contiguous pulses out of n pulses is of interest is in the performance analysis of antiradiation missiles.[1] These missiles search for a transmitting radar by detecting its presence using an electronic warfare receiver and then use this signal to guide it toward the

[1]The problem in this context was brought to the author's attention by E. J. O'Brien, manager of Hughes Aircraft Company's Surveillance and Sensor Systems Division.

transmitting radar. Since in general the pulse repetition frequency of the transmitting radar is unknown, one method of target acquisition is for the receiver of the missile to search for a periodic pulse train by looking for a periodic sequence of m successive pulses. For the purpose of antiradiation missile defense, it is of interest to the radar designer to determine at what range an antiradiation missile can first acquire the transmitting radar. A radar will typically transmit a waveform of n pulses in a particular direction while performing its search function. An antiradiation missile will generally be required to detect at least m of these pulses in a row in order to acquire the transmitting radar. For a fairly broad range of radar/missile receiver scenarios, the probability that any single pulse within the n-pulse waveform will be detected by the antiradiation missile receiver will be a constant p, and the event that any particular pulse within the waveform will be detected is statistically independent of the detection of any other pulse in the waveform. Hence the probability of acquisition by the antiradiation missile is equal to the probability of detecting *m or more* contiguous pulses in a pulse train of n pulses.

Finally, we note that there are situations in which the occurrence of too many contiguous occurrences of an event within some number of repeated trials can result in an undesirable situation. For example, in many digital communication systems, data synchronization is compromised if there are too many "0"s or too many "1"s in a row in a block of size n [4, ch. 10]. These channels are often referred to as runlength-constrained channels, and special codes have been developed for these types of channels [5, ch. 8]. Examples of such a channel include the magnetic recording channel and certain on–off keying (OOK) channels. In the case of the OOK channel, a block of too many consecutive 0s may result in a loss of symbol synchronization, assuming that 0 corresponds to the off state, as a long string of 0s appears to the receiver as a noise-only observation, providing no information about symbol stop and start times. The contiguous pulse detection analysis provides a method of calculating the probability of losing symbol synchronization in such systems when the data source can be modeled as a binary memoryless source.

The problem outlined above can be described mathematically as follows. Consider the combined experiment made up of n independent Bernoulli trials each having probability p of success. What is the probability that the resulting sequence will contain a sequence of m or more successes?

We would like to obtain a closed-form expression for this probability, yet despite the simplicity of the problem statement, a simple closed-form solution is not known. Asymptotic approximations to the solution of this problem have been developed [8], but in many cases of interest their accuracy is insufficient, and the only known way to analyze the problem is exhaustive enumeration of all possible success–failure patterns. In this work, we investigate two methods of solving this problem, one based on the powerful approach of generating functions, and the other based on a more straightforward recursive formula approach. We obtain closed-form solutions for the probability of the event of interest, as well as expressions for the mean and variance of the number of pulses n required for detection of m successive pulses. We also investigate the properties of statistical tests based on the detection of m successive pulses out of n pulses and note their properties with respect to related statistical tests. Finally, we comment on the application of these test to the antiradiation missile problem described above, and the performance of burst error correcting codes on memoryless channels.

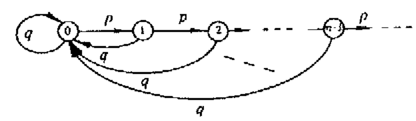

Fig. 1. State transition diagram of Markov chain describing number of successive detected pulses.

While not the same as the problem we are considering here, the related problem of calculating the probability that the moving average of the outcome of an independent sequence of Bernoulli trials exceeds a given threshold was considered by Brookner [6] and Dillard [7]. While both the problems of and the analytical techniques used in these two investigations differ from that of this work, the techniques of this work may be useful in the statistical analysis of the binary moving window detector considered in these two investigations.

II. PROBLEM FORMULATION

The problem being considered can be visualized using the state-transition diagram shown in Fig. 1. The state transition diagram consists of a directed graph with nodes corresponding to the states $0,1,2,\ldots,m$, representing the number of successive successful Bernoulli trials. If we are in state k (k successive detections), the probability we will be in state $k+1$ after the receiver observes the next pulse is p, and the probability we will return to state 0 is $q = 1 - p$. We note that this has the structure of a Markov chain, where the state corresponds to the number of successive successes. This Markov chain structure, as shown in Fig. 1, leads immediately to a directed graph which allows us to calculate generating functions for probabilities of interest in the solution of our problem using Mason's gain rule. Specifically, Fig. 1 as it is drawn allows us to calculate the probability of getting at least m successive successes for any number n trials using generating functions derived from a directed graph derived from the state transition diagram of the Markov chain. We now consider this approach.

Fig. 1 can be used to calculate the probability that m successive pulses are detected in an n pulse waveform. In order to see this, define

$$p_k^{(m)} = \Pr\{\text{state } m \text{ is first reached on } k\text{th transition}\}. \quad (1)$$

Clearly $p_k^{(m)} = 0$, for $k = 0,1,2,\ldots,m-1$. It also follows that if we define $p_m(n)$ as the probability that we reach state m at least once within n transitions, that

$$p_m(n) = \sum_{k=m}^{n} p_k^{(m)}. \quad (2)$$

Hence, solution of the problem requires a general expression for $p_k^{(m)}$, $m \le k \le n$, and $m \ge 1$. We now examine how to obtain such an expression.

III. PROBLEM SOLUTION

A. Generating Function Solution

A direct cambinatoric solution to this problem is difficult, and a solution could not be found in the literature. We solve the problem using generating functions [8]. The generating function $P_m(s)$ of the sequence of probabilities $\{p_0^{(m)}, p_1^{(m)}, \ldots, p_k^{(m)}, \ldots\}$ is defined as

$$P_m(s) = \sum_{k=0}^{\infty} p_k^{(m)} s^k. \quad (3)$$

Given the generating function $P_m(s)$, we can calculate the probabilities $p_k^{(m)}$ using the following relationship:

$$p_k^{(m)} = \frac{1}{k!} \frac{d^k P_m(s)}{ds^k}\bigg|_{s=0}. \quad (4)$$

An expression for the generating function $P_m(s)$ can be found using the theory of path enumeration in directed graphs, or equivalently, Mason's gain rule and the theory of signal-flow graphs [10, ch. 4]. We may redraw the state transition diagram of Fig. 1 as the directed graph of Fig. 2. Here the vertices are labeled 1 through m corresponding to the number of contiguous pulses detected, and the directed edges are labeled either ps or qs, corresponding to whether or not the next pulse was detected. The indeterminate s can be thought of as representing the unit delay between pulses, and hence the z-transform notation z^{-1} could have been used instead, but we chose to use s because it is the standard notational convention for generating functions [8, 9].

Let $G = (V, E)$ be the directed graph shown in Fig. 2, where $V = \{v_0, v_1, \ldots, v_m\}$ is the set of *vertices* or *nodes* of the graph, and E is a subset of ordered

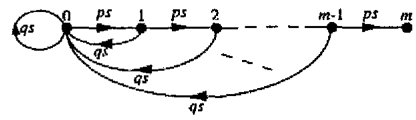

Fig. 2. Directed graph of Markov chain describing number of successive detected pulses.

pairs from V, called *edges*. Here, the set

$$E = \{(v_0,v_1),(v_1,v_2),\ldots,(v_{m-1},v_m),\\ (v_0,v_0),(v_1,v_0),\ldots,(v_{m-1},v_0)\}.$$

A *path* through the graph is a sequence of edges through the graph. A path can be specified by listing in order the vertices passed through while tracing the path through the graph. For example, one possible path from v_0 to v_2 is $v_0v_1v_0v_0v_1v_2$; another is simply $v_0v_1v_2$. A path is said to be of length k if exactly k edges are traversed in tracing out the path. An *open path* is a path along which no node is met twice. A *loop* is a closed path involving a set of nodes forming a complete circuit, for example $v_0v_1v_0$ or $v_0v_1v_2v_0$.

Now to each edge in G we assign a label, called the *transmittance* of the edge. Note that in the graph of Fig. 2, the label is ps if the edge leads from v_j to v_{j+1}, or qs if the edge leads from v_j to v_0, for $j = 0,1,\ldots,m-1$. We define the label or transmittance of a path through the graph as the product of the labels of the edged traversed in tracing out the path.

In order to determine $P_m(s)$, we must calculate the sum of the labels of all paths through the graph starting at v_0 and ending at v_m. We note that there are an infinite number of paths joining v_0 and v_m. There is one path of length m having label $(ps)^m$, one path of length $m+1$ having label $(qs)(ps)^m$, two paths of length $m+2$ having labels $(qs)^2(ps)^m$ and $qs(ps)^{m+1}$, and so on. In order to compute $P_m(s)$, we must calculate the sum of all the labels of all paths between v_0 and v_m. We will call this quantity the *transmittance* between v_0 and v_m and designate it T_{0m}.

An expression for the generating function $P_m(s)$ can be found by applying Mason's gain rule [10, 11], which states that the transmittance T_{xy} between node x and Node y is given by

$$T_{xy} = \frac{\sum_k T_k \Delta_k}{\Delta}. \quad (5)$$

Here T_k is the transmittance of the kth open path between x and y, Δ is the graph determinate, given by

$$\Delta = 1 - \sum_i L_i + \sum_{i,j} L_i' L_j' - \sum_{i,j,k} L_i'' L_j'' L_k'' + \cdots,$$

L_i is the loop transmittance of the ith loop, $L_i' L_j'$ is the product of the transmittances of two nontouching loops, with $\sum_{i,j} L_i' L_j'$, representing the sum of products of all pairs of loop transmittances of nontouching loops. Similarly $\sum_{i,j,k} L_i'' L_j'' L_k''$ represents the sum of the products of the loop transmittances of all combinations of three nontouching loops, and so forth. The quantity Δ_k is obtained by deleting from Δ any terms involving loops that have a node in common with a node in the kth path.

Applying Mason's Gain Rule to the graph of Fig. 2, we get

$$P_m(s) = \frac{(ps)^m}{1 - qs\sum_{k=0}^{m-1}(ps)^k} = \frac{(ps)^m}{1 - qs\dfrac{1-(ps)^m}{1-ps}} = \frac{p^m s^m(1-ps)}{1 - s + (1-p)p^m s^{m+1}}. \qquad (6)$$

Having an expression for the generating function, we can now evaluate the desired quantity $p_m(n)$ as

$$p_m(n) = \sum_{k=m}^{n}\left\{\frac{1}{k!}\frac{d^k P_m(s)}{ds^k}\bigg|_{s=0}\right\}. \qquad (7)$$

Note from (6) that for $p \in (0,1)$, $P_m(s)$ will always be analytic in some neighborhood containing $s = 0$. Hence derivatives of all orders exist at $s = 0$, so $p_m(n)$ as given in (7) is well defined.

For typical values of m and n, the expression given by (7) is very cumbersome to evaluate by hand. However, using a symbol manipulation program such as Mathematica [13] or Maple [14], this expression can be evaluated symbolically by machine. Its symbolic computation can be made more efficient by noting that if we define

$$\psi_{k,m}(s) = \frac{1}{k!}\frac{d^k P_m(s)}{ds^k} \qquad (8)$$

then we have the recursion,

$$\psi_{k,m}(s) = \frac{1}{k}\frac{d}{ds}\psi_{k-1,m}(s) \qquad (9)$$

and then (7) can be rewritten as

$$p_m(n) = \sum_{k=m}^{n}\psi_{k,m}(s)|_{s=0}. \qquad (10)$$

Although the generator function approach leads to large expressions which are algebraically cumbersome to work with, they provide the advantage of giving expressions for other quantities of interest as well as bounds on the probabilities of a number of events of interest. For example, the mean number of pulses K that need to be observed before m successive pulses are detected for the first time is given by

$$\overline{K} = \frac{dP_m(s)}{ds}\bigg|_{s=1} \qquad (11)$$

and the associated variance is

$$\sigma_K^2 = \left[\frac{d^2P_m(s)}{ds^2} + \frac{1}{s}\frac{dP_m(s)}{ds}\right]\bigg|_{s=1} - (\overline{K})^2.$$

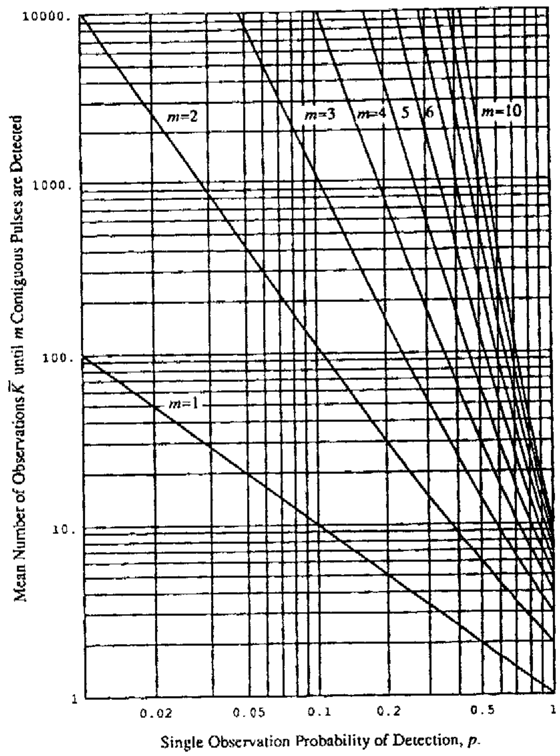

Fig. 3. Mean number of observations $\overline{K}$ required before m contiguous pulses are detected.

For $P_m(s)$ as given in (6) above, we have

$$\frac{dP_m(s)}{ds}\bigg|_{s=1} = \frac{1-p^m}{p^m(1-p)}$$

and

$$\frac{d^2P_m(s)}{ds^2}\bigg|_{s=1} = \frac{2(1 - 2p^m - mp^m + p^{2m} + p^{m+1} + mp^{m+1} - p^{2m+1})}{(1-p)^2p^{2m}}.$$

Thus

$$\overline{K} = \frac{1-p^m}{p^m(1-p)} \qquad (12)$$

and

$$\sigma_K^2 = \frac{1-(2m+1)p^m(1-p) - p^{2m+1}}{(1-p)^2p^{2m}}. \qquad (13)$$

Plots of $\overline{K}$ and the standard deviation σ_K can be found in Figs. 3 and 4. We note that $\overline{K}$ and σ_K are of interest in the antiradiation missile analysis problem described in Section I, and hence we see that these parameters are easily derived using generating functions.

Because $\overline{K}$ and σ_K^2 appear somewhat cryptic as given in (12) and (13), further insight into their behavior is provided by looking at approximate expressions for the cases $p \ll 1$ and $p \simeq 1$. For $p \ll 1$, it is straightforward to show that

$$\overline{K} \simeq \left(\frac{1}{p}\right)^m, \qquad p \ll 1 \qquad (14)$$

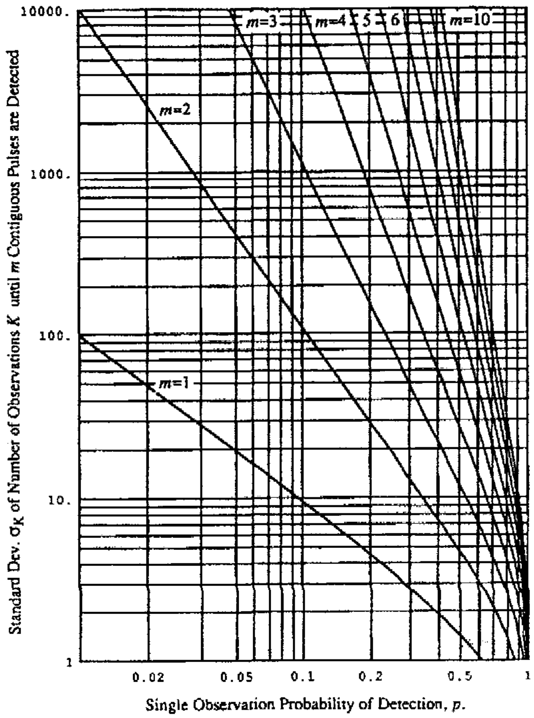

Fig. 4. Standard deviation σ_K of number of observations required before m contiguous pulses are detected.

and

$$\sigma_K^2 \simeq \left(\frac{1}{p}\right)^{2m}, \qquad p \ll 1. \tag{15}$$

For the case when $p \simeq 1$, we can obtain approximate expressions for $\overline{K}$ and σ_K^2 by expanding (12) and (13) as power series in p about $p = 1$. Doing so, we get

$$\overline{K} \simeq m + \frac{m(m+1)}{2}(1-p), \qquad p \simeq 1 \tag{16}$$

and

$$\sigma_K^2 \simeq \frac{m(m+1)(2m+1)}{6}(1-p), \qquad p \simeq 1. \tag{17}$$

For the case where $p \ll 1$, it is clear that $1/p \gg 1$, which implies that $\overline{K} = (1/p)^m$ will be very large—growing exponentially in m. A similar statement can be made about the variance $\sigma_K^2 = (1/p)^{2m}$. Note that both $\overline{K}$ and σ_K^2 grow without bound as $p \to 0$, as would be expected. For the case where $p \simeq 1$, we see from (16), that for $p = 1$, $\overline{K} = m$ with variance zero. This of course makes sense, because if a pulse is detected on each observation with probability one, then with probability one it will take exactly m observations to observe m contiguous pulse detections. As p becomes slightly less than one, $\overline{K}$ increases from m as expected, and the σ_K^2 becomes non-zero and proportional to $(1-p)$ for $p \simeq 1$.

Having the mean and variance of K, it is possible to bound the probability that K deviates more than some specified distance from the mean using the Chebyshev inequality. However, given the generating function $P_m(s)$, it is possible to generate a tighter and, for our purposes, more useful bound using the Chernoff bound. In order to do this, we define the *moment generating function* $\phi_K^{(m)}(t)$ of the number of trials K until m contiguous successes as

$$\phi_K^{(m)}(t) = E\{e^{tK}\} = \sum_{k=0}^{\infty} p_k^{(m)} e^{tk}$$

where t is real and E is the expectation operator. Then the Chernoff bound [15, 16, pp. 127–131], states that for any real number a,

$$\Pr\{K \geq a\} \leq e^{-ta}\phi_K^{(m)}(t), \qquad \forall \quad t \in (0, \gamma^+(m,p)) \tag{18}$$

and

$$\Pr\{K \leq a\} \leq e^{-ta}\phi_K^{(m)}(t), \qquad \forall \quad t < 0. \tag{19}$$

Here, the interval $(0, \gamma^+(m,p))$ in (18) is the intersection of the positive real numbers $(0, \infty)$ and the set $\{t : s = e^t$ is in the region of convergence the region of convergence of $P_m(s)$ about the origin$\}$. Equation (19) involves a similar minimization over $(\gamma^-(m,p), 0)$, but it can be shown that $\gamma^-(m,p) = -\infty$. In addition, by a straightforward extension of the argument of Section IIIC, where it is shown that the root s_1 of $P_m(s)$ with smallest modulus $|s_1|$ is real and lies in the interval $(0, 1/q)$ $(q = 1-p)$, it can be shown that $\gamma^+(m,p) \in (0, -\ln q)$.

These bounds hold for all t in the specified ranges ($t \in (0, \gamma^+(m,p))$ or negative t, respectively), however the tightest bounds are obtained by finding the t over the specified ranges that minimize the right-hand sides of these expressions. The right-hand sides of (18) and (19) are given by the expression

$$e^{-ta}\phi_K^{(m)}(t) = E\{e^{t(K-a)}\}$$

which has a value of 1 at $t = 0$, whose first derivative has a value of $\overline{K} - a$ at $t = 0$, and whose second derivative is positive for all t. So for $a > \overline{K}$, the right-hand side of (18) is greater than 1, and for $a < \overline{K}$, the right-hand side of (19) is greater than 1, neither case providing a useful bound. Hence it follows that we are only interested in (18) for the case where $a < \overline{K}$ and (19) for the case where $a > \overline{K}$.

Noting that

$$\phi_K^{(m)}(t) = P_m(s)|_{s=e^t}$$

we note that the tightest upper bounds of this form are given by

$$\Pr\{K \geq a\} \leq \min_{t \in (0, \gamma^+(m,p))} \left\{ \frac{e^{(m-a)t} p^m (1 - pe^t)}{1 - e^t + (1-p)p^m e^{(m+1)t}} \right\}, \qquad a > \overline{K} \tag{20}$$

and

$$\Pr\{K \leq a\} \leq \min_{t \in (-\infty, 0)} \left\{ \frac{e^{(m-a)t} p^m (1 - pe^t)}{1 - e^t + (1-p)p^m e^{(m+1)t}} \right\}, \qquad a < \overline{K}. \tag{21}$$

These expressions are useful in providing upper bounds on the probability that m contiguous successes have not or have occurred in a trials, where a is a positive integer such that $a > m$. In fact, these bounds may in some cases be sufficiently tight that they relieve the requirement for an exact solution of the problem. However, in many instances, an exact solution is still desired. The following example shows how the Chernoff bound can be used to bound probabilities of this kind.

EXAMPLE Using the Chernoff Bound, find an upper bound on the probability that 10 successive "heads" have not occurred within 10000 tosses of a fair coin.

In this case we use (20) to upper bound this probability. Using (12) with $m = 10$ and $p = 1/2$, we get $\overline{K} = 2043$, which is less than $a = 10000$, so (20) can provide a useful result. To determine $\gamma^+(m,p)$, we note that the expression to be maximized in (20) is monotonically decreasing in t and equal to one at $t = 0$. Because $(0, \gamma^+(m,p))$ is the intersection of the positive reals and the region of convergence, we can find $\gamma^+(m,p)$ by solving for the positive real root $s_0 = e^{\gamma^+(m,p)}$ with smallest absolute value of the denominator of $P_m(s)$. In this case, we solve

$$1 - s + \frac{1}{2048}s^{11} = 0$$

for the root $s_0 = 1.00049$, from which it follows that $\gamma^+(m,p) = 0.00049$. Hence because the expression in brackets is monotonically decreasing in t, the minimum occurs at $t_{\min} = \gamma^+(m,p) = 0.00049$. The minimum value achieved by this $t_{\min}$ is 0.00579. Hence we have

$$\Pr\{K > 10000\} \leq 0.00579.$$

B. Recursive Formula for Evaluating $p_k(n)$

While a simple combinatoric argument could not be found to give a closed-form solution for $p_k(n)$, it is possible to construct a recursive formula for the computation of $p_k(n)$. We can also calculate its generating function

$$Q_m(s) = \sum_{n=0}^{\infty} p_m(n)s^n$$

based on the recursive formula. The development is as follows.

If $p_m(n)$ is the probability of detecting m or more successive pulses out of a total of n pulses, we see that detecting m or more successive pulses in $n+1$ pulses can happen if and only if one of the following two disjoint events occur:

1) if m successive pulses were detected in the first n observations.

2) if the mth successive pulse detected occurred on the $n+1$ observation.

The probability of the first event is just $p_m(n)$. For the second event to be true, all three of the following events must be true.

1) In the first $n-m$ pulse observations, m or more successive pulses were not detected.

2) The $n-m+1$th pulse observation does not result in a detected pulse.

3) Pulse detections occurred in each of the last m pulse observations.

These three events are statistically independent, and have probabilities $1-p_m(n-m)$, $1-p$, and p^m, respectively. Hence it follows that

$$p_m(n+1) = p_m(n) + (1 - p_m(n-m))(1-p)p^m.$$

Of course for $n < m$, $p_m(n) = 0$, and for $n = m$, $p_m(n) = p^m$. Hence

$$p_m(n) = \begin{cases} 0, & \text{if} \quad n = 0,1,2,\ldots,m-1 \\ p^m, & \text{if} \quad n = m \\ p_m(n-1) + (1 - p_m(n-m-1))(1-p)p^m, & \text{if} \quad n > m. \end{cases}$$

A straightforward computation yields

$$\begin{aligned} Q_m(s) &= \sum_{n=m}^{\infty} p_m(n)s^n \\ &= \sum_{n=m}^{\infty}\left[\sum_{k=m}^{n} p_k^{(m)}\right]s^n \\ &= \frac{p^m s^m(1-ps)}{(1-s)(1-s+(1-p)p^m s^{m+1})}. \end{aligned} \quad (22)$$

Unfortunately, this generating function is no simpler to invert than $P_m(s)$, so a closed-form solution could not be obtained. The recursive formula itself is cumbersome to work with for large n and m, however using a symbol manipulation program, it is possible to get analytical results for specific m and n just as we can using $P_m(s)$. In fact, the results to be presented in Section IV were verified using both techniques.

C. An Asymptotic Expression for $p_k^{(m)}$

In principle, we can obtain an expression for $p_k^{(m)}$ by inverting the generating function $P_m(s)$. One approach to doing this is through use of a partial fraction expansion. We can write

$$P(s) = \frac{N(s)}{D(s)} = \frac{(ps)^m}{1 - qs\sum_{k=0}^{m-1}(ps)^k}$$

and then it follows that if $s_1, \ldots, s_m$ are the m distinct roots of $D(s)$, then we can write $P_m(s)$ as the partial fraction expansion

$$P_m(s) = \frac{\rho_1}{s_1 - s} + \frac{\rho_2}{s_2 - s} + \cdots + \frac{\rho_m}{s_m - s}.$$

Here the coefficients ρ_k are given by

$$\rho_k = \frac{-N(s_k)}{D'(s_k)}.$$

It then follows that upon inverting the partial fraction expansion of $P_m(s)$ term by term, we get

$$p_k^{(m)} = \frac{\rho_1}{s_1^{k+1}} + \frac{\rho_2}{s_2^{k+1}} + \cdots + \frac{\rho_m}{s_m^{k+1}}.$$

So assuming we can calculate the roots $s_1,\ldots,s_m$ of $D(s)$, and the corresponding coefficients $\rho_1,\ldots,\rho_m$, we have an exact expression for $p_k^{(m)}$. Unfortunately, calculation of all of these roots is difficult for large m and n. Generally, however, one term in this expression will dominate as n grows large. This is the term having the root s_k having the smallest modulus $|s_k|$. Thus if we label the roots $s_1,\ldots,s_m$ such that $|s_1| < |s_k|$, for $k = 2,\ldots,m$, we have the following asymptotic expression for $p_k^{(m)}$,

$$p_k^{(m)} \sim \frac{\rho_1}{s_1^{k+1}}. \tag{23}$$

(The notation $\sim$ indicates that the ratio of the left-hand side to the right-hand side approaches one as k grows large.) Of course, to make use of this result, we must solve for the single root s_1 of smallest modulus and evaluate the constant ρ_1.

If we consider the denominator

$$D(s) = 1 - qs(1 + ps + \cdots + p^{m-1}s^{m-1}),$$

we note that for positive, real s, $D(s)$ is a strictly decreasing function of s and that $D(0) = 1$ and $D(1/q) < 0$. Thus there exists a unique positive root s_1 of $D(s)$ on the real interval $(0, 1/q)$. Furthermore, for any real or complex s such that $|s| < s_1$,

$$\begin{aligned} &|qs(1 + ps + \cdots + p^{m-1}s^{m-1})| \\ &\quad \le qs_1(1 + ps_1 + \cdots + p^{m-1}s_1^{m-1}) \end{aligned}$$

with equality only when $s = s_1$. Thus s_1 is a positive real root of multiplicity one with $s_1 < |s_j|$ for $j = 2,\ldots,m$. This being the case, s_1 can easily be found numerically using any of a number of polynomial root-finding algorithms [17], and the asymptotic expression for $p_k(m)$ in (23) can be rewritten as

$$p_k^{(m)} \sim \frac{(s_1 - 1)(1 - ps_1)}{s_1^{k+1}(1 - p)(1 + m(1 - s_1))}. \tag{24}$$

Although these asymptotic approximations can be shown to be quite good for moderate to large k, we often need to evaluate $p_k^{(m)}$ for small k, especially when we are dealing with small m. Hence, these asymptotic expressions are not in general sufficient for an accurate estimate of $p_m(n) = \sum_{k=m}^{n} p_k^{(m)}$. For those situations where the approximation is sufficiently accurate, however, it can greatly simplify the analysis. A detailed analysis for the accuracy of the approximation in any particular case requires knowledge of p, m, and n.

IV. CONTIGUOUS PULSE BINARY INTEGRATION DETECTION PERFORMANCE

Because a general closed-form expression for the coefficients $p_k^{(m)}$ of $P_m(s)$ could not be found, we consider the specific cases of $P_1(s), P_2(s), \ldots, P_{10}(s)$. We can then find the $p_k^{(m)}$ in these cases using (3) and these $P_m(s)$. We can then find the $p_k(n)$ using (2). We consider the first ten terms in each case. This allows us to calculate the probability of detecting m or more contiguous pulses out of n total pulses for $1 \le k \le n$ and $n = 1,\ldots,10$ under the assumption that the detection of each individual pulse is an independent Bernoulli trial with probability p of success. The ten $P_m(s)$ are as follows:

$$\begin{aligned} P_1(s) &= ps + (1-p)ps^2 + (1-p)^2ps^3 \\ &\quad + (1-p)^3ps^4 + (1-p)^4ps^5 + (1-p)^5ps^6 \\ &\quad + (1-p)^6ps^7 + (1-p)^7ps^8 + (1-p)^8ps^9 \\ &\quad + (1-p)^9ps^{10} + O(s^{11}) \end{aligned}$$

$$\begin{aligned} P_2(s) &= p^2s^2 + (1-p)p^2s^3 + (1-p)p^2s^4 \\ &\quad + p^2(1 - p - p^2 + p^3)s^5 \\ &\quad + p^2(1 - p - 2p^2 + 3p^3 - p^4)s^6 \\ &\quad + (1-p)^3p^2(1+2p)s^7 \\ &\quad + (1-p)^3p^2(1 + 2p - p^2 - p^3)s^8 \\ &\quad + (1-p)^4p^2(1 + 3p + p^2 - p^3)s^9 \\ &\quad + (1-p)^4p^2(1 + 3p - 3p^3)s^{10} + O(s^{11}) \end{aligned}$$

$$\begin{aligned} P_3(s) &= p^3s^3 + (1-p)p^3s^4 + (1-p)p^3s^5 \\ &\quad + (1-p)p^3s^6 + p^3(1 - p - p^3 + p^4)s^7 \\ &\quad + p^3(1 - p - 2p^3 + 3p^4 - p^5)s^8 \\ &\quad + p^3(1 - p - 3p^3 + 5p^4 - 2p^5)s^9 \\ &\quad + p^3(1 - p - 4p^3 + 7p^4 - 3p^5)s^{10} + O(s^{11}) \end{aligned}$$

$$\begin{aligned} P_4(s) &= p^4s^4 + (1-p)p^4s^5 + (1-p)p^4s^6 + (1-p)p^4s^7 \\ &\quad + (1-p)p^4s^8 + p^4(1 - p - p^4 + p^5)s^9 \\ &\quad + p^4(1 - p - 2p^4 + 3p^5 - p^6)s^{10} + O(s^{11}) \end{aligned}$$

TABLE I
Coefficient Table for Polynomials $p_k(n)$

		Power of p									
k	n	1	2	3	4	5	6	7	8	9	10
1	1	1									
	2	2	-1								
	3	3	-3	1							
	4	4	-6	4	-1						
	5	5	-10	10	-5	1					
	6	6	-15	20	-15	6	-1				
	7	7	-21	35	-35	21	-7	1			
	8	8	-28	56	-70	56	-28	8	-1		
	9	9	-36	84	-126	126	-84	36	-9	1	
	10	10	-45	120	-210	252	-210	120	-45	10	-1
2	2		1								
	3		2	-1							
	4		3	-2	0						
	5		4	-3	-1	1					
	6		5	-4	-3	4	-1				
	7		6	-5	-6	9	-3	0			
	8		7	-6	-10	16	-5	-2	1		
	9		8	-7	-15	25	-6	-9	6	-1	
	10		9	-8	-21	36	-5	-24	18	-4	0
3	3			1							
	4			2	-1						
	5			3	-2	0					
	6			4	-3	0	0				
	7			5	-4	0	-6	1			
	8			6	-5	0	-3	4	-1		
	9			7	-6	0	-6	9	-3	0	
	10			8	-7	0	-10	16	-6	0	0

		Power of p						
k	n	4	5	6	7	8	9	10
4	4	1						
	5	2	-1					
	6	3	-2	0				
	7	4	-3	0	0			
	8	5	-4	0	0	0		
	9	6	-5	0	0	-1	1	
	10	7	-6	0	0	-3	4	-1
5	5		1					
	6		2	-1				
	7		3	-2	0			
	8		4	-3	0	0		
	9		5	-4	0	0	0	
	10		6	-5	0	0	0	0
6	6			1				
	7			2	-1			
	8			3	-2	0		
	9			4	-3	0	0	
	10			5	-4	0	0	0
7	7				1			
	8				2	-1		
	9				3	-2	0	
	10				4	-3	0	0
8	8					1		
	9					2	-1	
	10					3	-2	0
9	9						1	
	10						2	1
10	10							1

$$P_5(s) = p^5 s^5 + (1-p)p^5 s^6 + (1-p)p^5 s^7 + (1-p)p^5 s^8 + (1-p)p^5 s^9 + (1-p)p^5 s^{10} + O(s^{11})$$

$$P_6(s) = p^6 s^6 + (1-p)p^6 s^7 + (1-p)p^6 s^8 + (1-p)p^6 s^9 + (1-p)p^6 s^{10} + O(s^{11})$$

$$P_7(s) = p^7 s^7 + (1-p)p^7 s^8 + (1-p)p^7 s^9 + (1-p)p^7 s^{10} + O(s^{11})$$

$$P_8(s) = p^8 s^8 + (1-p)p^8 s^9 + (1-p)p^8 s^{10} + O(s^{11})$$

$$P_9(s) = p^9 s^9 + (1-p)p^9 s^{10} + O(s^{11})$$

$$P_{10}(s) = p^{10} s^{10} + O(s^{11}).$$

From (2), we know that the probability $p_m(m)$ of detecting m contiguous pulses within n pulses total is given by summing the generating function coefficients $p_k^{(m)}$ over K for $k = m,\ldots,n$. Hence, in general, $p_m(n)$ is an nth degree polynomial in p. Table I displays the coefficients of these polynomials for $p_1(n)$ through $p_{10}(n)$, for $n = 1,\ldots,10$. So, for example, the probability $p_4(9)$ of detecting four or more contiguous pulses out of six pulses total is given by (see Table I) as

$$p_4(9) = 6p^4 - 5p^5 - 1p^8 + 1p^9.$$

In order to examine the behavior of the probability of detecting m or more successive pulses out of n total pulses, we examine the case of $n = 10$. Here we have calculated all of the probabilities $p_k(10)$ using Table I.

The resulting polynomials are:

$$p_1(10) = 10p - 45p^2 + 120p^3 - 210p^4 + 252p^5 - 210p^6 + 120p^7 - 45p^8 + 10p^9 - p^{10}$$

$$p_2(10) = 9p^2 - 8p^3 - 21p^4 + 36p^5 - 5p^6 - 24p^7 + 18p^8 - 4p^9$$

$$p_3(10) = 8p^3 - 7p^4 - 10p^6 + 16p^7 - 6p^8$$

$$p_4(10) = 7p^4 - 6p^5 - 3p^8 + 4p^9 - p^{10}$$

$$p_5(10) = 6p^5 - 5p^6$$

$$p_6(10) = 5p^6 - 4p^7$$

$$p_7(10) = 4p^7 - 3p^8$$

$$p_8(10) = 3p^8 - 2p^9$$

$$p_9(10) = 2p^9 - p^{10}$$

$$p_{10}(10) = p^{10}.$$

The reader may note that it is really unnecessary to calculate $P_1(s)$ in the list of generating functions above or $p_1(n)$ in the list of probabilities $p_k(n)$ given above, as the probability of detecting one or more successive pulses out of n total pulses is just one minus the probability of detecting none. Hence $p_1(n) = 1 - (1-p)^n$. We include these expressions in the above lists for completeness. Note however that by use of the binomial theorem, it can be shown that $1-(1-p)^n$ is equal to the expression for $p_1(n)$ given in the list above.

Plots of $p_k(10)$ for $k = 1,\ldots,10$ as a function of p, the probability of detecting a single pulse, are shown in Fig. 5. This plot, plotted on logarithmic axes, explicitly shows the asymptotic behavior of these probabilities as p grows small. Note that p is plotted as a function of itself as a dotted line to serve as a line of reference for the behavior of the $p_k(10)$.

In considering the performance of contiguous pulse binary integration as a binary integration scheme for radar detection, it is useful to compare its performance with the more common binomial binary integration rule of "u or more detections out of n, regardless of order." We note that in this comparison, we are assuming that detection in each resolution cell is an independent Bernoulli trial with probability p. In situations where targets span many resolution cells, the contiguous integration approach may be more advantageous.

Thus binomial binary integration rule is easy to analyze since it just corresponds to u or more successes in n Bernoulli trials and can be written as [12, ch. 3]

$$p_B(u;n) = \sum_{j=u}^{n} \binom{n}{j} p^j (1-p)^{n-j}.$$

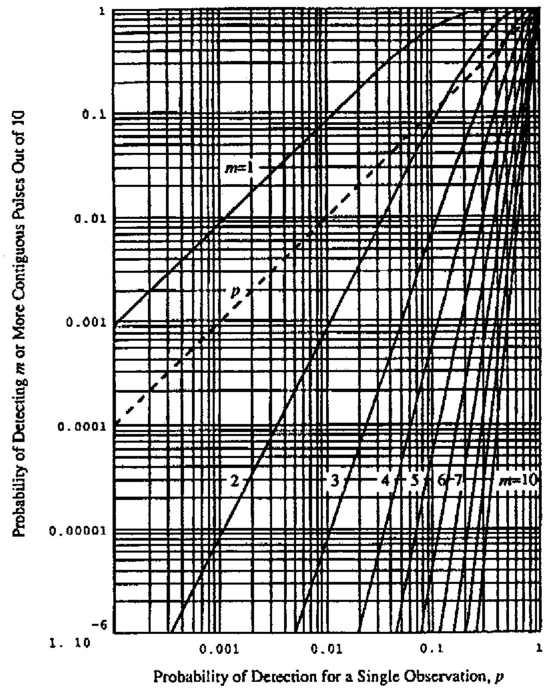

Fig. 5. Probability of detecting m or more contiguous pulses out of 10 pulses versus single observation probability of detection p.

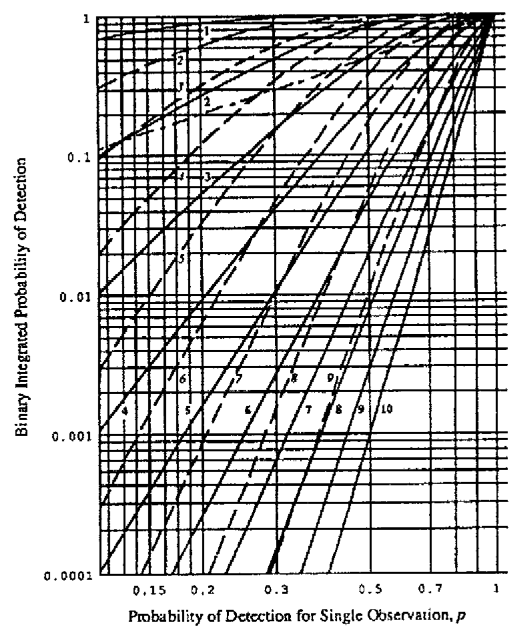

Fig. 6. Binary integrated probability of detection versus single observation probability of detection p. Dashed lines with italicized labels give probability of u or more detections out of 10, regardless of order. Solid lines with bold labels give probability of k or more contiguous detections out of 10 total.

Fig. 6 displays both $p_B(k;10)$ and $p_k(10)$ as a function of p in order to allow comparison of these two binary integration rules.

In general, we are interested in binary integration rules which increase overall detection probability over single observation detection probability p for moderate to large values of p, while decreasing overall detection probability below p for small values of p. The reason for this is that in most radar detection problems, there is a relatively small probability of declaring a target present when one is not—the probability of false alarm, P_{FA}—corresponding to a small probability of success in the single Bernoulli trial corresponding to the radar observation, while the probability of declaring a target present when one is in fact present—the probability of detection, P_D—corresponds to the probability of success in a Bernoulli trial with what is generally a much larger probability of success. Generally, we want to make the overall probability of false alarm as small as possible while making the probability of detection as large as possible, given the single observation probabilities of declaring a target present when one is not present or when one is not present, p_0, and declaring a target present when one is in fact present, p_1. As Fig. 6 indicates, at least for $n = 10$ pulses, $p_B(k;10)$ exhibits better binary integration characteristics than $p_k(10)$, even when selecting differing optimal values of k for the two quantities. Note that in general, $p_B(1;n) = p_1(n)$ and $p_B(n;n) = p_n(n)$, as indicated for $n = 10$ here. Again, we note that this is not surprising, because the ratio of the number of detections to the total number of pulses is a sufficient statistic for the single pulse probability of detection, and in general we can describe the composite hypotheses H_0 that no target is present and H_1 that a target is present in terms of the single pulse probability of detection p.

V. SUMMARY AND CONCLUSIONS

We have considered the problem of analyzing the probability of detecting m or more contiguous pulses out of a total of n pulses when it is assumed that the detection of each individual pulse is an independent Bernoulli trial with probability p. While a general closed-form expression would be ideal, one could not be found. Instead, we have presented an algorithm or procedure for constructing expressions for these probabilities as a function of p, m, and n. We have demonstrated the use of this procedure by explicitly presented results for the cases $m \leq n \leq 10$. We have also presented expressions and curves showing the mean and variance of the total number of pulses that need to be observed in order to get m contiguous detections when the probability of detection for each of the independent pulse detections is p. These results were obtained using the method of generating function, and the required generating functions were derived using Mason's Gain Rule. We then derived bounds on the probability of the events that the number of pulses K required for m successive successes of the form $\{K \geq a\}$ and $\{K \leq a\}$, where a is a positive number (most often a positive integer). These bounds were derived through use of the Chernoff bound, and involve only knowledge of the moment generating function $\phi_K^{(m)}(t)$, which in turn can be determined from knowledge of the generating function $P_m(s)$. These bounds demonstrate another benefit of the generator function approach, and in many cases, these bounds themselves may be sufficient to provide an approximate solution to the problem of interest.

Next we considered a solution based on a recursive formula for the probabilities $p_k(n)$. While this approach also led to an analytical procedure for finding the probabilities $p_m(n)$, it did not provide a simple way to calculate the statistics of the number of observations required until occurrence of m contiguous detections as the generating function approach did. Since these statistics are of interest in the antiradiation missile analysis described in Section I, it does not completely replace the generating function solution.

We then considered the derivation of asymptotic expressions for $p_k^{(m)}$ which can be used to calculate approximations for $p_k(n)$. This was done using the well-known technique of asymptotic expansions based on generating functions. While these approximations may be useful in some problems, they can be inaccurate for small k.

We then looked at the performance of contiguous pulse binary integration rules in detection problems, comparing them with the results one obtains using the simple unordered u out of n binary integration rule. As a typical case to examine in detail, we choose the case of $n = 10$ (10 successive observations) and consider the probability of getting k or more successive detections within the block as a function of the single pulse detection probability p, for $k = 1,\ldots,10$. In order to do this, we calculated the generating functions $P_1(s),\ldots,P_{10}(s)$, expanded each of them into a power series, and collected the first 10 (n) terms of each (note that the constant term of these polynomials will always be zero). From these expansions, the probabilities $p_k^{(m)}$ are extracted, and the probabilities $p_m(n)$ are calculated using the sum in (2). We note that while this was carried out for the case of $n = 10$, the process is easily generalized and implemented on a computer for other n. So in fact, analytical expressions can be generated for arbitrary m and n.

The generating function approach outlined in Section IIIA can be generalized to calculate the probability that any particular pattern of 0s and 1s occurs in a pattern of n bits. We have considered the contiguous pattern of m 1s in a Bernoulli trial generated bit stream, but patterns such as 1010101 or 0010111 could be just as easily considered. We note, however, that the resulting generating function will not have all of the same properties as those for the particular case of contiguous pulses that we investigated. So, for example, the development of asymptotic expressions for the probabilities of interest, as was done in Section IIIC may be quite different. Note, however, that once the generating functions are found, bounds on the probability of the occurrence of a particular bit sequence in a block of n bits can be easily constructed using the Chernoff bound argument of Section IIIA. Hence the technique outlined may be useful for the calculation of particular events occurring in repeated Bernoulli trials as they appear in a diverse collection of detection and communication problems.

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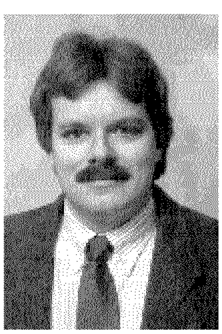

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