On the Commutativity of Discrete Memoryless Channels in Cascade

by MARK R. BELL

School of Electrical Engineering, Purdue University, West Lafayette, IN 47907-1285, U.S.A.

ABSTRACT: *The conditions are discussed under which two square discrete memoryless channels (DMCs) in cascade commute. Two different types of channel commutativity are considered:* matrix commutativity, *in which changing the order of two cascaded channels results in an identical overall channel, and* capacity commutativity, *in which the order of two cascaded channels results in an overall channel with the same capacity as the original cascade. A theorem is presented giving necessary and sufficient conditions for a pair of square DMCs to be matrix commutative and note its implications for a number of example channel cascades. Finally, it is shown that all pairs of r-ary symmetric channels are matrix commutative, regardless of their crossover probabilities.*

I. Introduction

Consider two binary symmetric channels, BSC_1 and BSC_2, with crossover probabilities ε_1 and ε_2, respectively. If we cascade these two channels such that BSC_2 follows BSC_1, the resulting channel is a binary symmetric channel with crossover probability

$$\varepsilon_{12} = \varepsilon_1(1-\varepsilon_2)+\varepsilon_2(1-\varepsilon_1). \tag{1}$$

If we exchange the order of cascade such that BSC_1 follows BSC_2, we get a binary symmetric channel with crossover probability

$$\varepsilon_{21} = \varepsilon_2(1-\varepsilon_1)+\varepsilon_1(1-\varepsilon_2). \tag{2}$$

The crossover probabilities of the BSCs resulting from the two possible cascades of BSC_1 and BSC_2 are equivalent, independent of the particular values of ε_1 and ε_2. Hence, we have that the order of two BSCs in a cascade is irrelevant, and we say that the two channels commute, providing an overall channel that is independent of their order.

Now consider two binary Z channels BZC_1 and BZC_2 with absorption probabilities δ_1 and δ_2, respectively. If we cascade these two channels with BZC_2 following BZC_1, the resulting channel is a binary Z channel with absorption probability

$$\delta_{12} = \delta_1+\delta_2-\delta_1\delta_2. \tag{3}$$

If we exchange the order of cascade such that BZC_1 follows BZC_2, we get a binary Z channel with absorption probability

$$\delta_{21} = \delta_2+\delta_1-\delta_2\delta_1. \tag{4}$$

Clearly these two cascades are equivalent as well, and we can say that BZC_1 and BZC_2 commute.

Now consider what happens when we cascade a binary symmetric channel with a binary Z channel. If we cascade BSC_1 with BZC_2 such that BZC_2 follows BSC_1, we get a binary discrete memoryless channel with crossover probabilities $\alpha_{12} = P(Y=1|X=0)$ and $\beta_{12} = P(Y=0|X=1)$ given by

$$\alpha_{12} = \varepsilon(1-\delta) \tag{5}$$

and

$$\beta_{12} = \varepsilon(1-\delta)+\delta. \tag{6}$$

If we form the cascade such that BSC_1 follows BZC_2, we get a binary discrete memoryless channel with crossover probabilities $\alpha_{21} = P(Y=1|X=0)$ and $\beta_{21} = P(Y=0|X=1)$ given by

$$\alpha_{21} = \varepsilon \tag{7}$$

and

$$\beta_{21} = \varepsilon(1-\delta)+\delta(1-\varepsilon). \tag{8}$$

These two cascades are *not* equivalent. In fact, the only time they can be equivalent is when at least one of the channels is a perfect binary channel (i.e. $\varepsilon = 0$ or $\delta = 0$). Hence we can say that a binary symmetric channel and a binary Z channel do not commute, while two binary symmetric channels or two binary Z channels will commute. This simple example begs the question: under what conditions will two *discrete memoryless channels* (DMCs) commute? In this paper, we investigate this question.

Historically, with a few exceptions, the problem of channels in cascade has been ignored. No work was found that discussed the conditions under which two channels in cascade commute.

The earliest paper discussing channels in cascade is that of Silverman (**1**). Silverman presents a detailed analysis of the general binary memoryless channel. He also investigates the capacity of length two cascades of binary memoryless channels. In particular, he notes that given a cascade of two identical binary symmetric channels and a cascade of two identical asymmetric binary channels, where all individual channels making up the cascades have identical capacities C_0, the capacity of the cascade of binary symmetric channels will be larger than that of the cascade of asymmetric channels *unless C_0 is sufficiently small.* He then considers this problem in detail for the case where the asymmetric channels are Z channels. Silverman's results are not easily extended to the case of DMCs with m inputs and m outputs, and he does not consider the conditions under which two channels commute.

Shannon (**2**) introduced a partial ordering of $r \times r$ DMCs that has some interesting implications for cascades of these channels. Shannon states that a channel with transition matrix Q_1 includes a channel with transition matrix Q_2 (written $Q_1 \supseteq Q_2$) if there exists another channel with transition matrix Q such that cascading the channels corresponding to Q_1 and Q results in a channel with transition matrix Q_2. An interesting property of Shannon's partial ordering relating to cascades is that if $Q_1 \supseteq Q_3$ and $Q_2 \supseteq Q_4$, then $Q_1Q_2 \supseteq Q_3Q_4$.

Simon (**3**) derives an expression for the capacity of a cascade of L identical $r \times r$ discrete memoryless channels in terms of the eigenvalues and eigenvectors of the constituent channel of the cascade when this channel has a nonsingular transition matrix. Although he applies linear algebraic ideas as we do, he does not consider the problem of when discrete memoryless channels will commute.

Posner and Rubin (**4**) consider the problem of a cascade of L binary symmetric channels in cascade, where they assume that the binary symmetric channels arise by making a binary decision on an additive white Gaussian noise channel with no bandwidth constraint. They show that for large L the capacity drops off by a factor asymptotic to $\ln L$.

Recently, a number of problems relating the capacity of a cascade of binary channels to the capacities of the individual channels making up the cascade have been considered (**5**). None of the above mentioned works, however, consider the conditions under which two discrete memoryless channels commute. In this correspondence, we consider this problem for square (r-input, r-output) discrete memoryless channels.

II. Mathematical Preliminaries

2.1. *Basic Definitions*

We now present some basic definitions relating to discrete memoryless channels in cascade. We follow the conventions and notation of McEliece (**6**).

A *discrete memoryless channel* (DMC) is characterized by two finite sets: the input alphabet A_x and the output alphabet A_y, and a set of transition probabilities $p(y|x)$ defined for each $x \in A_x$ and $y \in A_y$. If $|A_x| = r$ and $|A_y| = s$, we can take $A_x = \{0, \ldots, r-1\}$ and $A_y = \{0, \ldots, s-1\}$. We can characterize the DMC by an $r \times s$ stochastic matrix Q with $[Q]_{x,y} = p(y/x)$, known as the *transition matrix* of the channel. A *square* DMC is a DMC for which $r = s$, or equivalently $A_x = A_y$. In this case, Q is an $r \times r$ square matrix. Note that if $\mathbf{p}_x = (p_1, \ldots, p_r)$ is the vector of probabilities of inputs to the DMC and $\mathbf{p}_y = (q_1, \ldots, q_s)$ is the vector of probabilities of outputs from the DMC, then $\mathbf{p}_y = \mathbf{p}_x Q$.

A DMC A with transition matrix Q_A is said to be *matrix equivalent* to a DMC B with transition matrix Q_B if $Q_A = Q_B$. A DMC A with matrix Q_A is said to be *row-permutation equivalent* to a DMC B with matrix Q_B if there exists a permutation π of the rows of Q_B such that $Q_A = \pi(Q_B)$. Similarly, a DMC A is said to be *column-permutation equivalent* to a DMC B if there exists a permutation ρ of the columns of Q_B such that $Q_A = \rho(Q_B)$. A DMC A is said to be *permutationally matrix equivalent* to a DMC B if there exists a permutation of the rows π and a permutation of the columns ρ such that $Q_A = \pi(\rho(Q_B))$ [or equivalently, $Q_A = \rho(\pi(Q_B))$]. Note that the significance of permutational matrix equivalence is that row permutation

corresponds to a relabeling of the inputs and a column permutation corresponds to a relabeling of the outputs. Such relabelings do not effect the capacity of the channel.

Two DMCs A and B with matrices Q_A and Q_B, respectively, are said to be $A \supset B$ *cascadable* if the matrix product Q_AQ_B is well defined. The resulting channel $A \supset B$ has channel transition matrix

$$Q_{A \supset B} = A_AQ_B.$$

Two cascadable square channels A and B are said to be *matrix commutative* if

$$Q_AQ_B = Q_BQ_A,$$

or equivalently, if $A \supset B$ and $B \supset A$ are matrix equivalent. Two cascadable square channels A and B are said to be *capacity commutative* if

$$\text{cap}\,(Q_{A \supset B}) = \text{cap}\,(Q_{B \supset A}).$$

Here cap (Q) denotes the capacity of the DMC with transition matrix Q. Note that for any row permutation π and column permutation ρ,

$$\text{cap}\,(\pi(\rho(Q))) = \text{cap}\,(Q).$$

Thus it follows that matrix commutativity is a stronger condition than capacity commutativity.

2.2. *Results from matrix theory*

We now present some definitions and results from matrix theory that will be useful.

Two $n \times n$ matrices Q_A and Q_B are said to be *similar* if there exists a nonsingular matrix S such that $Q_A = SQ_BS^{-1}$. If an $n \times n$ matrix Q is similar to a diagonal matrix, then Q is said to be *diagonalizable*. Two diagonalizable $n \times n$ matrices Q_A and Q_B are said to be *simultaneously diagonalizable* if there is a single similarity matrix S such that both SQ_AS^{-1} and SQ_BS^{-1} are diagonal.

The following theorem provides a well known result for diagonalizable matrices.

Theorem I

Let Q be an $n \times n$ matrix. Then Q is diagonalizable if and only if there is a set of n linearly independent eigenvectors of Q.

Proof: See (**7**, p. 46). ■

Theorem I tells us that an $n \times n$ diagonalizable matrix defined over the field F provides a complete eigenvector basis for F^n. This leads us to the main theorem that allows us to prove our results.

Theorem II

If Q_A and Q_B are diagonalizable $n \times n$ matrices, they commute if and only if they have a complete set of eigenvectors in common.

Proof: If Q_A and Q_B commute, then for any nonsingular matrix S, so do SQ_AS^{-1}

and SQ_BS^{-1}. So commutation does not depend on the particular basis chosen. Suppose we select a basis such that

$$Q'_A = SQ_AS^{-1} = \operatorname{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \ldots),$$

where I_{n_i} is an identity matrix of dimension n_i, the multiplicity of λ_i. The Q_B must have the form

$$Q'_B = SQ_BS^{-1} = \operatorname{diag}(B_1, B_2, \ldots),$$

where B_i has dimension $n_i \times n_i$ (the same dimension as I_{n_i}). Now define Q''_A and Q''_B on a new basis defined by the transformation T such that

$$Q''_A = TQ'_AT^{-1} = (TS)Q_A(TS)^{-1},$$

and

$$Q''_B = TQ'_BT^{-1} = (TS)Q_B(TS)^{-1},$$

where T is of the form

$$T = \operatorname{diag}(T_1, T_2, \ldots)$$

with T_i having dimension $n_i \times n_i$. Then Q''_A can be written as

$$Q''_A = \operatorname{diag}(\lambda_1 T_1 T_1^{-1}, \lambda_2 T_2 T_2^{-1}, \ldots) = Q'_A.$$

So $Q''_A = Q'_A$ regardless of $T_1, T_2, \ldots$. On the other hand

$$Q''_B = \operatorname{diag}(T_1 B_1 T_1^{-1}, T_2 B_2 T_2^{-1}, \ldots).$$

Since Q_B and hence Q'_B are diagonalizable, $B_1, B_2, \ldots$ are also diagonalizable. Hence, we can choose $T_1, T_2, \ldots$ to diagonalize $B_1, B_2, \ldots$, and hence Q''_B. So we can find a basis that simultaneously diagonalizes Q_A and Q_B. It follows that the rows of (TS) are a linearly independent set of simultaneous eigenvectors for both Q_A and Q_B.

Conversely, assume that $\{\mathbf{x}_i\}$ is a complete common set of eigenvectors for both Q_A and Q_B. Then any $\mathbf{y} \in F^n$ can be written as

$$\mathbf{y} = \sum_{i=0}^{n} a_i x_i.$$

If $\mathbf{x}_i Q_A = \lambda_i \mathbf{x}_i$ and $\mathbf{x}_i Q_B = \mu_i \mathbf{x}_i$, then for all $\mathbf{y} \in F^n$,

$$\mathbf{y} Q_A Q_B = \left(\sum_{i=0}^{n} a_i \mathbf{x}_i Q_A\right) Q_B = \left(\sum_{i=0}^{n} a_i \lambda_i \mathbf{x}_i\right) Q_B = \sum_{i=0}^{n} a_i \lambda_i \mu_i \mathbf{x}_i,$$

and

$$\mathbf{y} Q_B Q_A = \left(\sum_{i=0}^{n} a_i \mathbf{x}_i Q_B\right) Q_A = \left(\sum_{i=0}^{n} a_i \mu_i \mathbf{x}_i\right) Q_A = \sum_{i=0}^{n} a_i \mu_i \lambda_i \mathbf{x}_i.$$

Thus $Q_A Q_B \mathbf{y} = Q_B Q_A \mathbf{y}$ for all $\mathbf{y} \in F^n$, and hence $Q_A Q_B = Q_B Q_A$. ■

Commutativity of diagonalizable Q_A and Q_B implies that they are simultaneously diagonalizable.

III. Results on Commuting Discrete Channels

Theorem II provides us with the necessary result to prove our main theorem on the matrix commutativity channels with a diagonalizable transition matrix.

Theorem III

If A and B are discrete memoryless channels with diagonalizable channel transition matrices Q_A and Q_B, respectively, then A and B are matrix commutative if and only if Q_A and Q_B have a complete set of eigenvectors in common.

Proof: The proof is a straightforward application of Theorem II. A and B will be matrix commutative if and only if $Q_A Q_B = Q_B Q_A$, but by Theorem II, this is true if and only if Q_A and Q_B have a complete set of eigenvectors in common. ■

Theorem III provides a straightforward test to determine whether or not two $r \times r$ DMCs with diagonalizable transition matrices are matrix commutative. The major limitation on the channels handled by this theorem is that both channels have diagonalizable transition matrices. As we will see, many, if not most of the channels of interest have diagonalizable transition matrices. We now look at several examples of the application of this theorem. We will first address the examples noted in Section I.

Example 1: *Matrix commutativity of BSCs*. As we noted in Section I, any two binary symmetric channels are matrix commutative, regardless of their crossover probabilities. According to Theorem III, if the transition matrix

$$Q_\varepsilon = \begin{pmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{pmatrix}$$

is diagonalizable, then since any two BSCs are matrix commutative, all BSCs must have two linearly independent eigenvectors that are independent of the crossover probability ε. The eigenvalues of Q_ε are $\lambda_1 = 1$ and $\lambda_2 = 1-2\varepsilon$. The corresponding eigenvectors are $e_1 = (1, 1)$ and $e_2 = (-1, 1)$. Clearly, the BSC has a complete set of linearly independent eigenvectors that are independent of the crossover probabilities, so by Theorem III, any two BSCs are matrix commutative, as we know from Section I.

Example 2: *Matrix commutativity of BZCs*. As we noted in Section I, any two binary Z channels are matrix commutative, regardless of their absorption probabilities. According to Theorem III, if the transition matrix

$$Q_\delta = \begin{pmatrix} 1 & 0 \\ \delta & 1-\delta \end{pmatrix}$$

is diagonalizable, then since any two BZCs are matrix commutative, all BZCs must have two linearly independent eigenvectors that are independent of the absorption probability δ. The eigenvalues of Q_δ are $\lambda_1 = 1$ and $\lambda_2 = 1-\delta$. The corresponding eigenvectors are $e_1 = (1,0)$ and $e_2 = (-1,1)$. Clearly, the BZC has a complete set of linearly independent eigenvectors that are independent of the crossover probabilities, so by Theorem III, any two BZCs are matrix commutative, as we know from Section I.

Example 3 : *A BSC and a BZC are not matrix commutative*. As we demonstrated in Section I, a BSC and a BZC are *not* matrix commutative. That this must be the case can immediately be seen from Theorem III and the results of the previous two examples. Both the BSC and BZC have diagonalizable transition matrices, but don't have common eigenvectors, with the exception of the trivial case where either the BSC or the BZC is a perfect binary channel ($\varepsilon = 0$ or $\delta = 0$).

Using Theorem III, it is easy to deduce the conditions under which two binary memoryless channels are matrix commutative. We present these conditions in the following theorem.

Theorem IV

Two binary memoryless channels with transition matrices

$$Q_1 = \begin{pmatrix} 1-\alpha_1 & \alpha_1 \\ \beta_1 & 1-\beta_1 \end{pmatrix}$$

and

$$Q_2 = \begin{pmatrix} 1-\alpha_2 & \alpha_2 \\ \beta_2 & 1-\beta_2 \end{pmatrix}$$

are matrix commutative if and only if $\beta_1/\alpha_1 = \beta_2/\alpha_2$.

Proof: A general binary memoryless channel with transition matrix

$$Q_{\alpha,\beta} = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1-\alpha-\beta$, with corresponding eigenvectors $e_1 = (\beta/\alpha, 1)$ and $e_2 = (-1,1)$. Thus by Theorem III, two channels (α_1, β_1) and (α_2, β_2) are matrix commutative if and only if they have a complete set of eigenvectors in common. This is true if and only if $\beta_1/\alpha_1 = \beta_2/\alpha_2$. ■

Using Muroga's Square-Channel Capacity Theorem (**8**, **9**), the capacity of a discrete binary channel with transition matrix

$$Q_{\alpha,\beta} = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

can be shown to be (**1**)

$$C(\alpha,\beta) = \frac{\beta\mathscr{H}_2(\alpha)-(1-\alpha)\mathscr{H}_2(\beta)}{1-(\alpha+\beta)} + \log_2\left[1+2^{(\mathscr{H}_2(\beta)-\mathscr{H}_2(\alpha))/(1-(\alpha+\beta))}\right] \quad \text{(bits)}. \qquad (9)$$

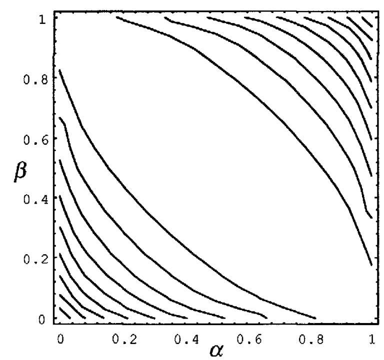

FIG. 1. Contours of equal capacity for the binary memoryless channel.

In order to examine the results of this theorem for the binary channel, we use a diagram introduced by Silverman in (**1**). This diagram, shown in Fig. 1, displays contours of equal capacity as a function of α and β.

Superimposing lines of constant β/α—lines along which all channels are matrix commutative—we get the diagram of Fig. 2. Note the line with $\beta/\alpha = 1$ along which all BSCs lie and the vertical line $\alpha = 0$ along which all Z channels lie.

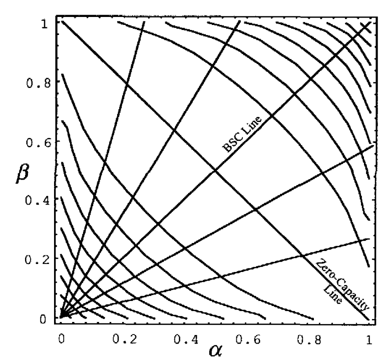

FIG. 2. Lines of constant β/α, along which binary memoryless channels commute.

We now investigate the matrix commutativity of two other symmetric channels. An r-*ary symmetric channel* is defined as a channel with an $r \times r$ transition matrix Q with

$$[Q]_{x,y} = \begin{cases} \varepsilon, & \text{if } x \neq y, \\ 1-(r-1)\varepsilon, & \text{if } x = y, \end{cases}$$

for $0 \leqslant \varepsilon \leqslant 1/(r-1)$ (**6**, p. 55). We now have the following results for the ternary symmetric channel ($r = 3$) and the quaternary symmetric channel ($r = 4$).

Theorem V

Any two ternary symmetric channels are matrix commutative, regardless of their crossover probabilities.

Proof: The channel transition matrix of a ternary symmetric channel with crossover probabilities ε is

$$Q_3 = \begin{pmatrix} 1-2\varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1-2\varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1-2\varepsilon \end{pmatrix}, \quad 0 \leqslant \varepsilon \leqslant 1/2.$$

The eigenvalues of this matrix are $\lambda_1 = 1$, $\lambda_2 = 1-3\varepsilon$ and $\lambda_3 = 1-3\varepsilon$; the corresponding eigenvectors are $e_1 = (1,1,1)$, $e_2 = (-1,0,1)$ and $e_3 = (-1,1,1)$. Hence, Q_3 is a diagonalizable matrix with a complete set of linearly independent eigenvectors which are independent of the crossover probability ε. So by Theorem III, any two ternary symmetric channels are matrix commutative. ■

Theorem VI

Any two quaternary symmetric channels are matrix commutative, regardless of their crossover probabilities.

Proof: The channel transition matrix of a quaternary symmetric channel with crossover probabilities ε is

$$Q_4 = \begin{pmatrix} 1-3\varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1-3\varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1-3\varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1-3\varepsilon \end{pmatrix}, \quad 0 \leqslant \varepsilon \leqslant 1/3.$$

The eigenvalues of this matrix are $\lambda_1 = 1$, $\lambda_2 = 1-4\varepsilon$, $\lambda_3 = 1-4\varepsilon$ and $\lambda_4 = 1-4\varepsilon$; the corresponding eigenvectors are $e_1 = (1,1,1,1)$, $e_2 = (-1,0,0,1)$, $e_3 = (-1,0,1,0)$ and $e_4 = (-1,1,0,0)$. Hence, Q_4 is a diagonalizable matrix with a complete set of linearly independent eigenvectors that are independent of the crossover probability ε. So by Theorem III, any two ternary symmetric channels are matrix commutative. ■

The results for the r-ary symmetric channel for $r = 3$ and $r = 4$ leads us to

conjecture that any two r-ary symmetric channels are matrix commutative for arbitrary r. In fact, this is true and we present the result in the following theorem.

Theorem VII

Any two r-ary symmetric channels are matrix commutative, regardless of their crossover probabilities.

Proof: We could use Theorem III to prove this result, but in this case it is easier to examine $Q_r(\varepsilon_1)Q_r(\varepsilon_2)$ and $Q_r(\varepsilon_2)Q_r(\varepsilon_1)$ directly. The channel transition matrix of an r-ary symmetric channel with crossover probabilities ε is

$$Q_r(\varepsilon) = \begin{pmatrix} 1-(r-1)\varepsilon & \varepsilon & \dots & \varepsilon \\ \varepsilon & 1-(r-1)\varepsilon & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon & \epsilon & \dots & 1-(r-1)\varepsilon \end{pmatrix}, \quad 0 \leqslant \varepsilon \leqslant 1/(r-1).$$

It is easy to verify that for $0 \leqslant \varepsilon_1, \varepsilon_2 \leqslant 1/(r-1)$, $Q_r(\varepsilon_1)Q_r(\varepsilon_2) = Q_r(\varepsilon_2)Q_r(\varepsilon_1)$, since

$$[Q_r(\varepsilon_1)Q_r(\varepsilon_2)]_{x,y} = [Q_r(\varepsilon_2)Q_r(\varepsilon_1)]_{x,y} = \begin{cases} \varepsilon_1+\varepsilon_2-r\varepsilon_1\varepsilon_2, & \text{if } x \neq y, \\ 1-(r-1)[\varepsilon_1+\varepsilon_2-r\varepsilon_1\varepsilon_2], & \text{if } x = y. \end{cases}$$

Hence, by definition, any two r-ary symmetric channels are matrix commutative. ■

IV. Summary and Conclusions

We have considered the conditions under which two square discrete memoryless channels, both having diagonalizable transition matrices, in cascade commute. Actually, we have defined two different kinds of commutativity, matrix commutativity and capacity commutativity, and considered in detail the stronger of the two, matrix commutativity. We then proved a theorem stating the conditions under which two diagonalizable channels in cascade commute, obtaining the result that they commute if and only if their transition matrices have a complete set of eigenvectors in common and are hence simultaneously diagonalizable. We then applied these results to a number of examples in which channels in cascade are matrix commutative and are not matrix commutative. In particular, we derived the conditions under which general binary memoryless channels are matrix commutative, and we also showed that all cascades of two r-ary symmetric channels are matrix commutative, regardless of their crossover probabilities. These results can be easily extended to cascades of more than two channels by noting that simultaneous diagonalizability generates an equivalence class of diagonalizable matrices.

References

(**1**) R. A. Silverman, "On binary channels and their cascades", *IRE Trans. Inform. Theory*, Vol. 30, No. 3, pp. 19–27, 1955.

(2) C. E. Shannon, "A note on partial ordering for communication channels", *Inform. Control*, Vol. 1, pp. 390–398, 1958.
(3) M. K. Simon, "On the capacity of a cascade of identical discrete memoryless nonsingular channels", *IEEE Trans. Inform. Theory*, Vol. 16, No. 1, pp. 100–102, 1970.
(4) E. C. Posner and A. L. Rubin, "The capacity of digital links in tandem", *IEEE Trans. Inform. Theory*, Vol. 30, pp. 464–470, 1984.
(5) A. B. Kiely, J. T. Coffey and M. R. Bell, "On the capacity of channels in cascade", IEEE Int. Symp. on Information Theory, 24–26 June 1991, Bundapest, Hungary.
(6) R. L. McEliece, "The Theory of Information and Coding", Addison-Wesley, Reading, MA, 1977.
(7) R. A. Horn and C. R. Johnson, "Matrix Analysis", Cambridge University Press, New York, NY, 1985.
(8) S. Muroga, "On the capacity of a discrete channel, Part I", *J. Physical Soc. Japan*, Vol. 8, No. 4, pp. 484–494, 1953.
(9) Robert B. Ash, "Information Theory", Wiley Interscience, New York, 1965.

Received: 13 March 1993
Accepted: 24 May 1993