

Dataflow Analysis

Wednesday, November 4, 2009

Program optimizations

- So far we have talked about different kinds of optimizations
 - Peephole optimizations
 - Local common sub-expression elimination
 - Loop optimizations
- What about *global optimizations*
 - Optimizations across multiple basic blocks (usually a whole procedure)
 - Not just a single loop

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Useful optimizations

- Common subexpression elimination (global)
 - Need to know which expressions are available at a point
- Dead code elimination
 - Need to know if the effects of a piece of code are never needed, or if code cannot be reached
- Constant folding
 - Need to know if variable has a constant value
- Loop invariant code motion
 - Need to know where and when variables are live
- So how do we get this information?

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Dataflow analysis

- Framework for doing compiler analyses to drive optimization
- Works across basic blocks
- Examples
 - Constant propagation: determine which variables are constant
 - Liveness analysis: determine which variables are live
 - Available expressions: determine which expressions are have valid computed values
 - Reaching definitions: determine which definitions could “reach” a use

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Example: constant propagation

- Goal: determine when variables take on constant values
- Why? Can enable many optimizations
 - Constant folding

```
x = 1;  
y = x + 2;  
if (x > z) then y = 5  
... y ...
```

→

- Create dead code

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Example: constant propagation

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```

x = 1;
y = x + 2;
if (x > z) then y = 5
... y ...
    
```

→

```

x = 1;
y = 3;
if (x > z) then y = 5
... y ...
    
```

- Create dead code

```

x = 1;
y = x + 2;
if (y > x) then y = 5
... y ...
    
```

→

```

x = 1;
y = 3; //dead code
if (true) then y = 5 //simplify!
... y ...
    
```

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How can we find constants?

- Ideal: run program and see which variables are constant
- Problem: variables can be constant with some inputs, not others – need an approach that works for all inputs!
- Problem: program can run forever (infinite loops?) – need an approach that we know will finish fairly quickly
- Idea: run program *symbolically*
- Essentially, keep track of whether a variable is constant or not constant (but nothing else)

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Overview of algorithm

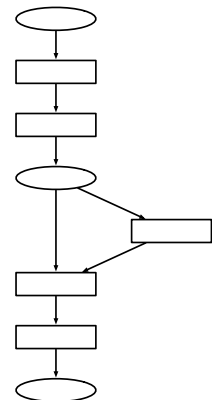
- Build control flow graph
 - We'll use statement-level CFG (with merge nodes) for this
- Perform symbolic evaluation
 - Keep track of whether variables are constant or not
- Replace constant-valued variable uses with their values, try to simplify expressions and control flow

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Build CFG

```

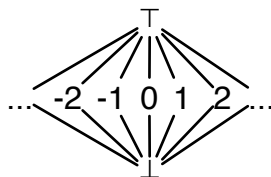
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Symbolic evaluation

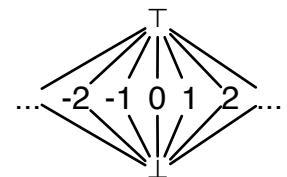
- Idea: replace each value with a symbolic
 - constant (specify which), maybe constant, definitely not constant
- Can organize these possible values in a *lattice* (will formalize this later)



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Symbolic evaluation

- Evaluate expressions symbolically: $eval(e, V_{in})$
- If e evaluates to a constant, return that value. If any input is \top (or \perp), return \top (or \perp)
 - Why?
- Two special operations on lattice
 - $meet(a, b)$ – highest value less than or equal to both a and b
 - $join(a, b)$ – lowest value greater than or equal to both a and b

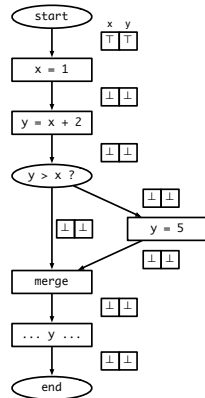


Join often written as $a \sqcup b$
Meet often written as $a \sqcap b$

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Putting it together

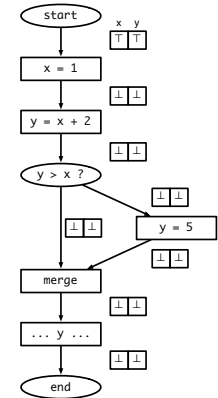
- Keep track of the symbolic value of a variable at every program point (on every CFG edge)
- State vector
- What should our initial value be?
 - Starting state vector is all \perp
 - Can't make any assumptions about inputs – must assume not constant
- Everything else starts as \perp , since we don't know if the variable is constant or not at that point



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Executing symbolically

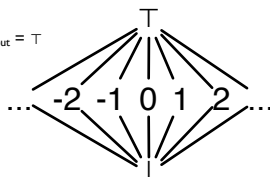
- For each statement $t = e$ evaluate e using V_{in} , update value for t and propagate state vector to next statement
- What about switches?
 - If e is true or false, propagate V_{in} to appropriate branch
 - What if we can't tell?
 - Propagate V_{in} to both branches, and symbolically execute both sides
- What do we do at merges?



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Handling merges

- Have two different V_{in} s coming from two different paths
- Goal: want new value for V_{in} to be **safe** (shouldn't generate wrong information), and we don't know which path we actually took
- Consider a single variable. Several situations:
 - $V_1 = \perp, V_2 = * \rightarrow V_{out} = *$
 - $V_1 = \text{constant } x, V_2 = x \rightarrow V_{out} = x$
 - $V_1 = \text{constant } x, V_2 = \text{constant } y \rightarrow V_{out} = \perp$
 - $V_1 = \perp, V_2 = * \rightarrow V_{out} = \perp$
- Generalization:
 - $V_{out} = V_1 \sqcup V_2$



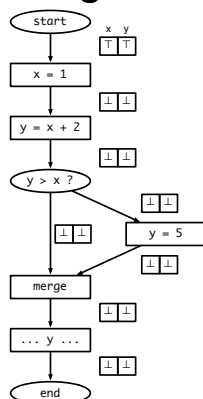
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Result: worklist algorithm

- Associate state vector with each edge of CFG, initialize all values to \perp , worklist has just start edge
- While worklist not empty, do:
 - Process the next edge from worklist
 - Symbolically evaluate target node of edge using input state vector
 - If target node is assignment ($x = e$), propagate $V_{in}[\text{eval}(e)/x]$ to output edge
 - If target node is branch (e ?)
 - If $\text{eval}(e)$ is true or false, propagate V_{in} to appropriate output edge
 - Else, propagate V_{in} along both output edges
 - If target node is merge, propagate $\text{join}(\text{all } V_{in})$ to output edge
 - If any output edge state vector has changed, add it to worklist

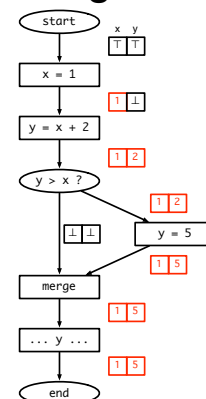
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Running example



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Running example



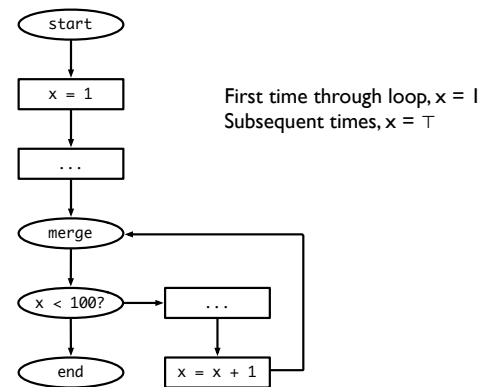
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What do we do about loops?

- Unless a loop never executes, symbolic execution looks like it will keep going around to the same nodes over and over again
- Insight: if the input state vector(s) for a node don't change, then its output doesn't change
 - If input stops changing, then we are done!
- Claim: input will eventually stop changing. Why?

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Loop example



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Complexity of algorithm

- $V = \#$ of variables, $E = \#$ of edges
- Height of lattice = 2 \rightarrow each state vector can be updated at most $2 * V$ times.
- So each edge is processed at most $2 * V$ times, so we process at most $2 * E * V$ elements in the workload.
- Cost to process a node: $O(V)$
- Overall, algorithm takes $O(EV^2)$ time

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Question

- Can we generalize this algorithm and use it for more analyses?
- First, let's lay the theoretical foundation for dataflow analysis.

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Lattice Theory

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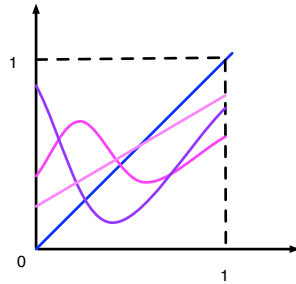
First, something interesting

- **Brouwer Fixpoint Theorem**
 - Every continuous function f from a closed disk into itself has at least one fixed point
- More formally:
 - Domain D : a *convex, closed, bounded* subspace in a plane (generalizes to higher dimensions)
 - Function $f: D \rightarrow D$
 - There exists some x such that $f(x) = x$

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Intuition

- Consider the one-dimensional case: mapping a line segment onto itself
- $x \in [0, 1]$
- $f(x) \in [0, 1]$
- There must exist some x for which $f(x) = x$
- Examples (in 2D)
 - A mall directory
 - Crumpling up a piece of graph paper



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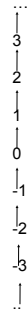
Back to dataflow

- Game plan:
 - Finite partially ordered set with least element: D
 - Function $f: D \rightarrow D$
 - Monotonic function $f: D \rightarrow D$
 - \exists fixpoint of f
 - \exists least fixpoint of f
 - Generalization to case when D has a greatest element, \top
 - \exists greatest fixpoint of f
 - Generalization to systems of equations

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Partially ordered set (poset)

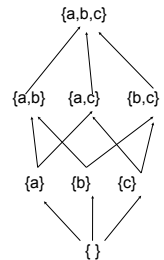
- Set D with a relation \sqsubseteq that is
 - Reflexive: $x \sqsubseteq x$
 - Anti-symmetric: $x \sqsubseteq y$ and $y \sqsubseteq x \Rightarrow y = x$
 - Transitive: $x \sqsubseteq y, y \sqsubseteq z \Rightarrow x \sqsubseteq z$
- Example: set of integers and \leq
- Graphical representation of poset
 - Graph in which nodes are elements of D and relation \sqsubseteq is indicated by arrows
 - Usually omit reflexive and transitive arrows for legibility
 - Not counting reflexive edges, graph is always a DAG (why?)



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Another example

- Powerset of any set, ordered by \subseteq is a poset
- In the example, poset elements are $\{\}, \{a\}, \{a, b\}, \{a, b, c\}$, etc.
- $X \sqsubseteq Y$ iff $X \subseteq Y$



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Finite poset with least element

- Poset in which
 - Set is finite
 - There is a least element that is below all other elements in poset
- Examples
 - Set of integers ordered by \leq is *not* a finite poset with least element (no least element, not finite)
 - Set of natural numbers ordered by \leq has a least element (0), but not finite
 - Set of factors of 12, ordered by \leq has a least element as is finite
 - Powerset example from before is finite (how many elements?) with a least element ($\{\}$)

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Domains

- "Finite poset with least element" is a mouthful, so we will abbreviate this to *domain*
- Later, we will add additional conditions to domains that are of interest to us in the context of dataflow analysis
 - (Goal: what is a lattice?)

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Functions on domains

- If D is a domain, we can define a function $f: D \rightarrow D$
 - Function maps each element of domain on to another element of the domain
- Example: for $D = \text{powerset of } \{a, b, c\}$
 - $f(x) = x \cup \{a\}$
 - $g(x) = x - \{a\}$
 - $h(x) = \{a\} - x$

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Monotonic functions

- A function $f: D \rightarrow D$ on a domain D is *monotonic* if
 - $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$
- Note: this is not the same as $x \sqsubseteq f(x)$
 - This means that x is *extensive*
- Intuition: think of f as an electrical circuit mapping input to output
 - If f is monotonic, raising the input voltage raises the output voltage (or keeps it the same)
 - If f is extensive, the output voltage is always the same or more than the input voltage

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Examples

- Domain D is the powerset of $\{a, b, c\}$
- Monotonic functions:
 - $f(x) = \{ \}$ (why?)
 - $f(x) = x \cup \{a\}$
 - $f(x) = x - \{a\}$
- Not monotonic
 - $f(x) = \{a\} - x$ (why?)
- Extensivity
 - $f(x) = x \cup \{a\}$ is monotonic *and* extensive
 - $f(x) = x - \{a\}$ is monotonic but not extensive
 - $f(x) = \{a\} - x$ is neither
- What is a function that is extensive, but not monotonic?

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Fixpoints

- Suppose $f: D \rightarrow D$.
 - A value x is a *fixpoint* of f if $f(x) = x$
 - f maps x to itself
- Examples: D is a powerset of $\{a, b, c\}$
 - Identity function: $f(x) = x$
 - Every element is a fixpoint
 - $f(x) = x \cup \{a\}$
 - Every set that contains a is a fixpoint
 - $f(x) = \{a\} - x$
 - No fixpoints

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Fixpoint theorem

- One form of *Knaster-Tarski Theorem*:
If D is a domain and $f: D \rightarrow D$ is monotonic, then f has at least one fixpoint
- More interesting consequence:
If \perp is the least element of D , then f has a *least fixpoint*, and that fixpoint is the largest element in the chain
 $\perp, f(\perp), f(f(\perp)), f(f(f(\perp))) \dots f^n(\perp)$
- Least fixpoint: a fixpoint of f , x such that, if y is a fixpoint of f , then $x \sqsubseteq y$

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Examples

- For domain of powersets, $\{ \}$ is the least element
- For identity function, $f^n(\{ \})$ is the chain
 $\{ \}, \{ \}, \{ \}, \dots$ so least fixpoint is $\{ \}$, which is correct
- For $f(x) = x \cup \{a\}$, we get the chain
 $\{ \}, \{a\}, \{a\}, \dots$ so least fixpoint is $\{a\}$, which is correct
- For $f(x) = \{a\} - x$, function is not monotonic, so not guaranteed to have a fixpoint!
- Important observation: as soon as the chain repeats, we have found the fixpoint (why?)

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Proof of fixpoint theorem

- First, prove that largest element of chain $f^n(\perp)$ is a fixpoint
- Second, prove that $f^n(\perp)$ is the *least* fixpoint

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Solving equations

- If D is a domain and $f: D \rightarrow D$ is a monotone function on that domain, then the equation $f(x) = x$ has a least fixpoint, given by the largest element in the sequence
 $\perp, f(\perp), f(f(\perp)), f(f(f(\perp))) \dots$
- Proof follows directly from fixpoint theorem

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Adding a top

- Now let us consider domains with an element \top , such that for every point x in the domain, $x \sqsubseteq \top$
- New theorem: if D is a domain with a greatest element \top and $f: D \rightarrow D$ is monotonic, then the equation $x = f(x)$ has a *greatest* solution, and that solution is the smallest element in the sequence
 $\top, f(\top), f(f(\top)), \dots$
- Proof?

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Multi-argument functions

- If D is a domain, a function $f: D \times D \rightarrow D$ is monotonic if it is monotonic in each argument when the other is held constant
- Intuition:
 - Electrical circuit has two inputs
 - If you raise either input while holding the other constant, the output either goes up or stays the same

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Fixpoints of multi-arg functions

- Can generalize fixpoint theorem in a straightforward way
- If D is a domain and $f, g: D \times D \rightarrow D$ are monotonic, the following system of equations has a least fixpoint solution, calculated in the obvious way
 $x = f(x, y)$ and $y = g(x, y)$
- Can generalize this to more than two variables and domains with greatest elements easily

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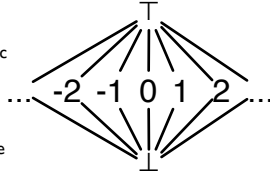
Lattices

- A bounded *lattice* is a partially ordered set with a \perp and \top , with two special functions for any pair of points x and y in the lattice:
 - A *join*: $x \sqcup y$ is the least element that is greater than x and y (also called the *least upper bound*)
 - A *meet*: $x \sqcap y$ is the greatest element that is less than x and y (also called the *greatest lower bound*)
- Are \sqcup and \sqcap monotonic?

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More about lattices

- Bounded lattices with a finite number of elements are a special case of domains with \top (why are they not the same?)
- Systems of monotonic functions (including \sqcup and \cap) will have fixpoints
- But some lattices are infinite! (example: the lattice for constant propagation)
 - It turns out that you can show a monotonic function will have a least fixpoint for any lattice (or domain) of *finite height*
 - Finite height: any totally ordered subset of domain (this is called a *chain*) must be finite
 - Why does this work?



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Solving system of equations

- Consider

$$x = f(x, y, z)$$

$$y = g(x, y, z)$$

$$z = h(x, y, z)$$
- Obvious iterative solution: evaluate every function at every step:

$$\perp \quad f(\perp, \perp, \perp) \quad \dots$$

$$\perp \quad g(\perp, \perp, \perp) \quad \dots$$

$$\perp \quad h(\perp, \perp, \perp) \quad \dots$$

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Worklist algorithm

- Obvious point: only necessary to re-evaluate functions whose inputs have changed
- Worklist algorithm
 - Initialize worklist with all equations
 - Initialize solution vector S to all \perp
 - While worklist not empty
 - Get equation from worklist
 - Re-evaluate equation based on S , update entry corresponding to lhs in S
 - Put all equations which use this lhs on their rhs in the worklist
- Claim: the worklist algorithm for constant propagation is an instance of this approach

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Mapping worklist algorithm

- Careful: the "variables" in constant propagation are not the individual variable values in a state vector. Each variable (from a fixpoint perspective) is an entire state vector – there are as many variables as there are edges in the CFG
- Functions:
 - Program statements: $\text{eval}(e, V_{in})$
 - These are called *transfer functions*
 - Need to make sure this is monotonic
 - Branches
 - Propagates input state vector to output – trivially monotonic
 - Merges
 - Use join or meet to combine multiple input variables – monotonic by definition

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Constant propagation

- Step 1: choose lattice
 - Use constant lattice (infinite, but finite height)
- Step 2: choose direction of dataflow
 - Run forward through program
- Step 3: create monotonic transfer functions
 - If input goes from \perp to constant, output can only go up. If input goes from constant to \top , output goes to \top
- Step 4: choose *confluence operator*
 - What do do at merges? For constant propagation, use join

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