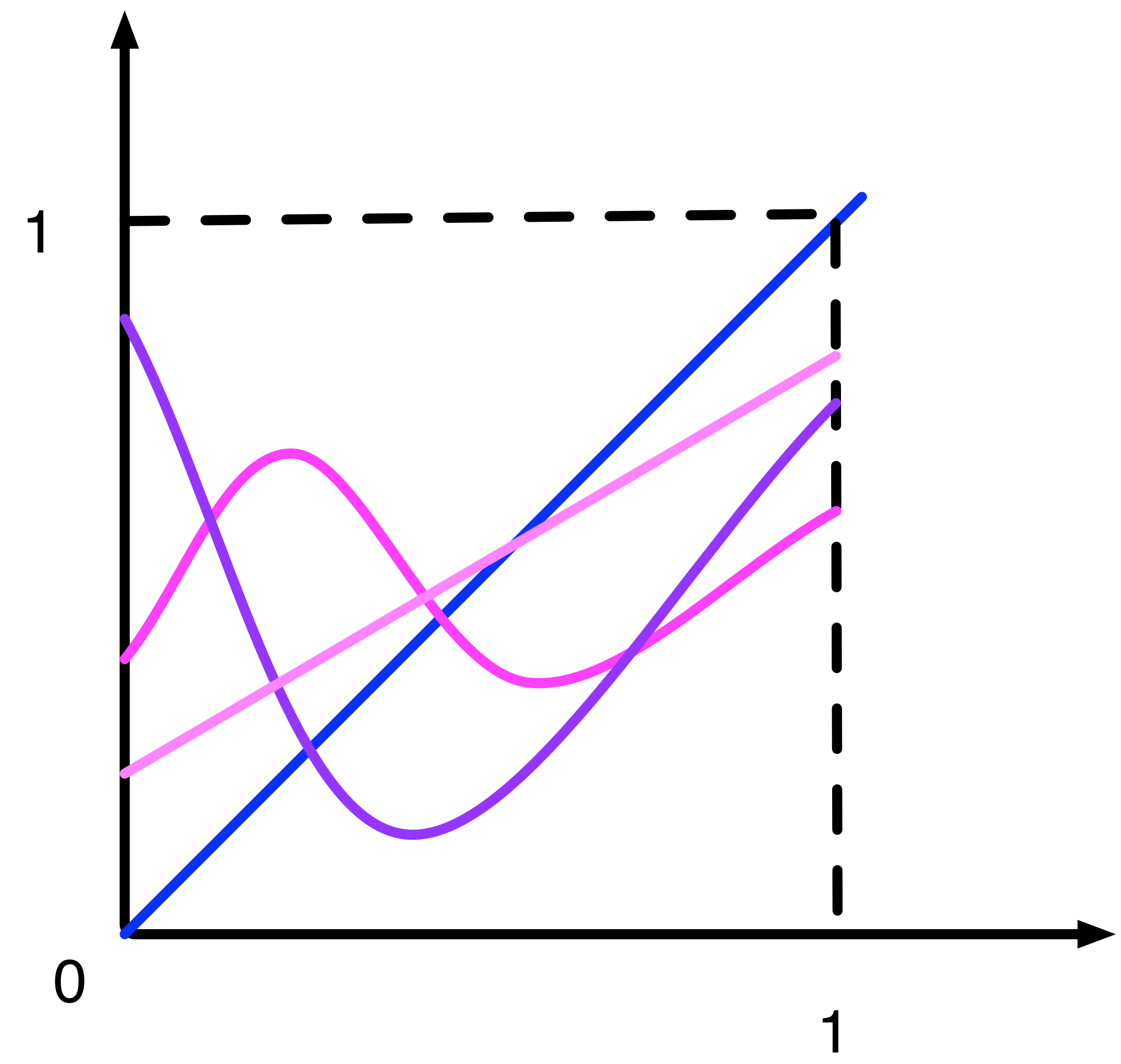
Lattice Theory

First, something interesting

- Brouwer Fixed point Theorem
  - Every continuous function *f* from a closed disk into itself has at least one fixed point
- More formally:
  - Domain D: a *convex, closed, bounded* subspace in a plane (generalizes to higher dimensions)
  - Function $f : D \rightarrow D$
  - There exists some x such that $f(x) = x$

Intuition

- Consider the one-dimensional case: mapping a line segment onto itself
- $x \in [0, 1]$
- $f(x) \in [0, 1]$
- There must exist some x for which $f(x) = x$
- Examples (in 2D)
  - A mall directory
  - Crumpling up a piece of graph paper

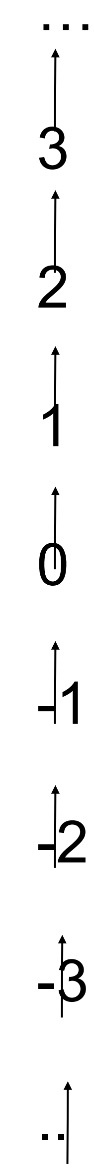


Back to dataflow

- Game plan:
  - Finite partially ordered set with least element: D
  - Function $f : D \to D$
  - Monotonic function $f : D \to D$
  - $\exists$ fixpoint of f
    - $\exists$ *least* fixpoint of f
  - Generalization to case when D has a greatest element, $\top$
    - $\exists$ *greatest* fixpoint of f
  - Generalization to systems of equations

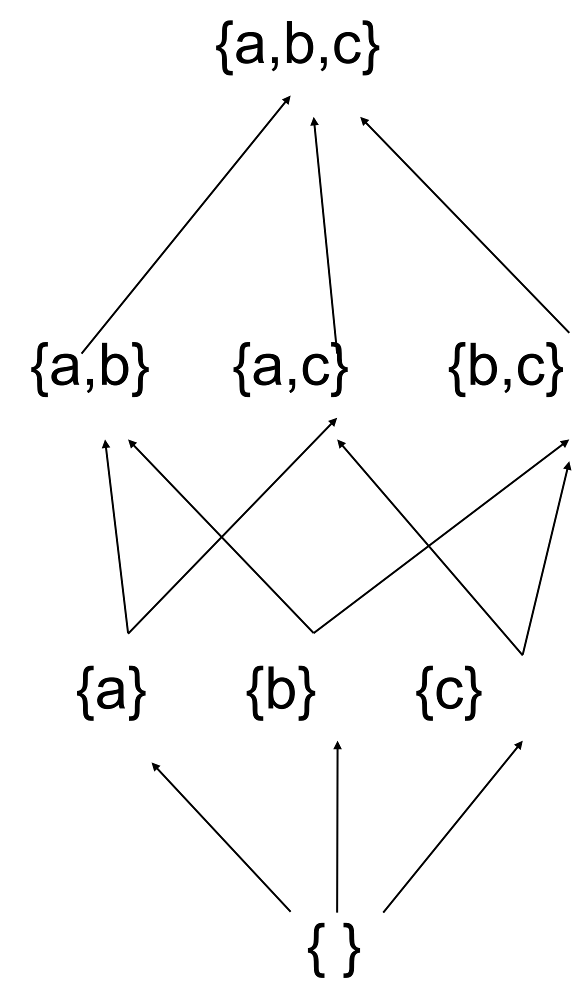
Partially ordered set (poset)

- Set *D* with a relation ⊑ that is
  - Reflexive: x ⊑ x
  - Anti-symmetric: x ⊑ y and y ⊑ x ⇒ y = x
  - Transitive: x ⊑ y, y ⊑ z ⇒ x ⊑ z
- Example: set of integers and ≤
- Graphical representation of poset
  - Graph in which nodes are elements of *D* and relation ⊑ is indicated by arrows
  - Usually omit reflexive and transitive arrows for legibility
  - Not counting reflexive edges, graph is always a DAG (why?)



Another example

- Powerset of any set, ordered by ⊆ is a poset
- In the example, poset elements are {}, {a}, {a, b}, {a, b, c}, etc.
- X ⊑ Y iff X ⊆ Y



Finite poset with least element

- Poset in which
  - Set is finite
  - There is a least element that is below all other elements in poset
- Examples
  - Set of integers ordered by ≤ is *not* a finite poset with least element (no least element, not finite)
  - Set of natural numbers ordered by ≤ has a least element (0), but not finite
  - Set of factors of 12, ordered by ≤ has a least element as is finite
  - Powerset example from before is finite (how many elements?) with a least element ({ })

Domains

- “Finite poset with least element” is a mouthful, so we will abbreviate this to *domain*
- Later, we will add additional conditions to domains that are of interest to us in the context of dataflow analysis
  - (Goal: what is a lattice?)

Functions on domains

- If D is a domain, we can define a function $f : D \rightarrow D$
  - Function maps each element of domain on to another element of the domain
- Example: for D = powerset of {a, b, c}
  - f(x) = x ∪ {a}
  - g(x) = x – {a}
  - h(x) = {a} – x

Monotonic functions

- A function $f : D \rightarrow D$ on a domain D is *monotonic* if
  - $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$
- Note: this is not the same as $x \sqsubseteq f(x)$
  - This means that x is *extensive*
- Intuition: think of f as an electrical circuit mapping input to output
  - If f is monotonic, raising the input voltage raises the output voltage (or keeps it the same)
  - If f is extensive, the output voltage is always the same or more than the input voltage

Examples

- Domain D is the powerset of {a, b, c}
- Montonic functions:
  - f(x) = { } (why?)
  - $f(x) = x \cup \{a\}$
  - $f(x) = x - \{a\}$
- Not monotonic
  - $f(x) = \{a\} - x$ (why?)
- Extensivity
  - $f(x) = x \cup \{a\}$ is monotonic *and* extensive
  - $f(x) = x - \{a\}$ is monotonic but not extensive
  - $f(x) = \{a\} - x$ is neither
- What is a function that is extensive, but not monotonic?

Fixpoints

- Suppose $f : D \rightarrow D$.
  - A value x is a *fixpoint* of f if $f(x) = x$
  - f maps x to itself
- Examples: D is a powerset of {a, b, c}
  - Identity function: f(x) = x
    - Every element is a fixpoint
  - $f(x) = x \cup \{a\}$
    - Every set that contains a is a fixpoint
  - $f(x) = \{a\} - x$
    - No fixpoints

Fixpoint theorem

- One form of *Knaster-Tarski Theorem*:

  If *D* is a domain and *f* : *D* → *D* is monotonic, then *f* has at least one fixpoint

- More interesting consequence:

  If ⊥ is the least element of *D*, then *f* has a *least fixpoint*, and that fixpoint is the largest element in the chain

  $\bot, f(\bot), f(f(\bot)), f(f(f(\bot))) \ldots f^n(\bot)$

- Least fixpoint: a fixpoint of *f*, x such that, if y is a fixpoint of *f*, then $x \sqsubseteq y$

Examples

- For domain of powersets, { } is the least element
- For identity function, $f^n(\{\ \})$ is the chain

  { }, { }, { }, ... so least fixpoint is { }, which is correct
- For $f(x) = x \cup \{a\}$, we get the chain

  { }, {a}, {a}, ... so least fixpoint is {a}, which is correct
- For $f(x) = \{a\} - x$, function is not monotonic, so not guaranteed to have a fixpoint!
- Important observation: as soon as the chain repeats, we have found the fixpoint (why?)

Proof of fixpoint theorem

- First, prove that largest element of chain $f^n(\perp)$ is a fixpoint

- Second, prove that $f^n(\perp)$ is the *least* fixpoint

Solving equations

- If D is a domain and $f : D \rightarrow D$ is a monotone function on that domain, then the equation f(x) = x has a least fixpoint, given by the largest element in the sequence

  $\perp$, f($\perp$), f(f($\perp$)), f(f(f($\perp$))) ...

- Proof follows directly from fixpoint theorem

Adding a top

- Now let us consider domains with an element $\top$, such that for every point x in the domain, $x \sqsubseteq \top$
- New theorem: if *D* is a domain with a greatest element $\top$ and $f : D \to D$ is monotonic, then the equation x = f(x) has a *greatest* solution, and that solution is the smallest element in the sequence

  $\top$, f($\top$), f(f($\top$)), ...
- Proof?

Multi-argument functions

- If D is a domain, a function $f : D \times D \rightarrow D$ is monotonic if it is monotonic in each argument when the other is held constant
- Intuition:
  - Electrical circuit has two inputs
  - If you raise either input while holding the other constant, the output either goes up or stays the same

Fixpoints of multi-arg functions

- Can generalize fixpoint theorem in a straightforward way
- If D is a domain and $f, g : D \times D \to D$ are monotonic, the following system of equations has a least fixpoint solution, calculated in the obvious way

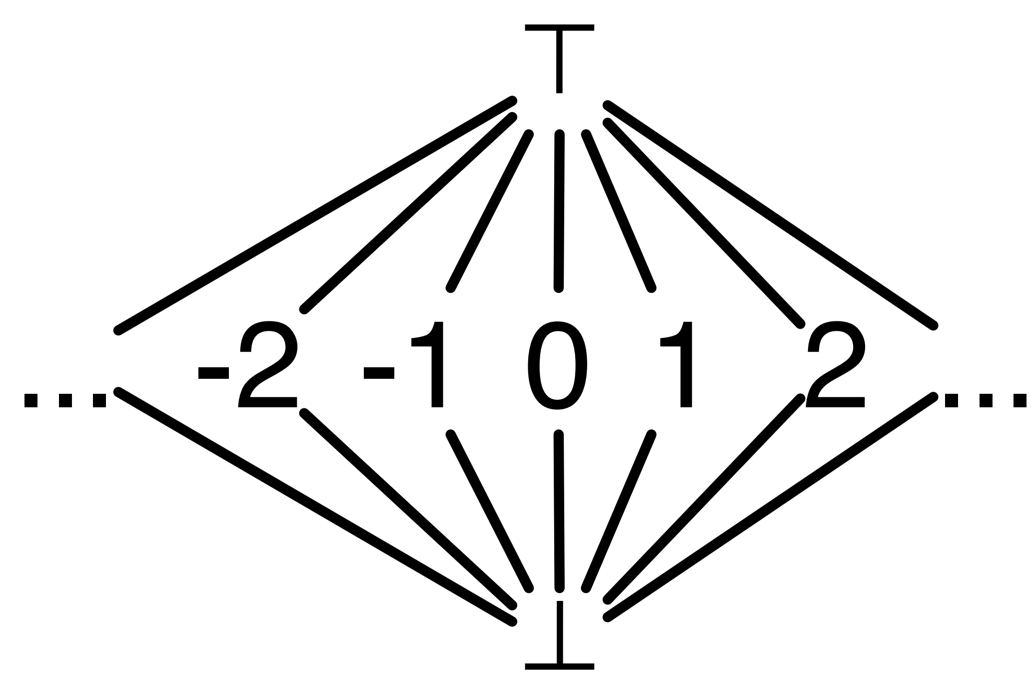
  $x = f(x, y)$ and $y = g(x, y)$
- Can generalize this to more than two variables and domains with greatest elements easily

Lattices

- A bounded *lattice* is a partially ordered set with a $\bot$ and $\top$, with two special functions for any pair of points *x* and *y* in the lattice:
    - A *join*: x $\sqcup$ y is the least element that is greater than x and y (also called the *least upper bound*)
    - A *meet*: x $\sqcap$ y is the greatest element that is less than x and y (also called the *greatest lower bound*)
- Are $\sqcup$ and $\sqcap$ monotonic?

More about lattices

- Bounded lattices with a finite number of elements are a special case of domains with ⊤ (why are they not the same?)
  - Systems of monotonic functions (including ⊔ and ⊓) will have fixpoints
- But some lattices are infinite! (example: the lattice for constant propagation)
  - It turns out that you can show a monotonic function will have a least fixpoint for any lattice (or domain) of *finite height*
  - Finite height: any totally ordered subset of domain (this is called a *chain*) must be finite
  - Why does this work?



Solving system of equations

- Consider

  x = f(x, y, z)

  y = g(x, y, z)

  z = h(x, y, z)

- Obvious iterative solution: evaluate every function at every step:

  $\perp$ $f(\perp,\perp,\perp)$ ...

  $\perp$ $g(\perp,\perp,\perp)$ ...

  $\perp$ $h(\perp,\perp,\perp)$ ...

Worklist algorithm

- Obvious point: only necessary to re-evaluate functions whose "important" inputs have changed
- Worklist algorithm
  - Initialize worklist with all equations
  - Initialize solution vector S to all $\perp$
  - While worklist not empty
    - Get equation from worklist
    - Re-evaluate equation based on S, update entry corresponding to lhs in S
    - Put all equations which use this lhs on their rhs in the worklist
- Claim: the worklist algorithm for constant propagation is an instance of this approach

Mapping worklist algorithm

- Careful: the "variables" in constant propagation are not the individual variable values in a state vector. Each variable (from a fixpoint perspective) is an entire state vector – there are as many variables as there are edges in the CFG
- Functions:
  - Program statements: eval(e,V_{in})
    - These are called *transfer functions*
    - Need to make sure this is monotonic
  - Branches
    - Propagates input state vector to output – trivially monotonic
  - Merges
    - Use join or meet to combine multiple input variables – monotonic by definition

Constant propagation

- Step 1: choose lattice
  - Use constant lattice (infinite, but finite height)
- Step 2: choose direction of dataflow
  - Run forward through program
- Step 3: create monotonic transfer functions
  - If input goes from $\perp$ to constant, output can only go up. If input goes from constant to $\top$, output goes to $\top$
- Step 4: choose *confluence operator*
  - What do do at merges? For constant propagation, use join