# Lattice Theory



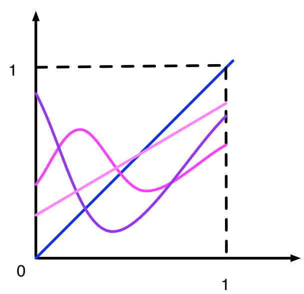
# First, something interesting

- Brouwer Fixed point Theorem
  - Every continuous function *f* from a closed disk into itself has at least one fixed point
- More formally:
  - Domain $D$: a *convex, closed, bounded* subspace in a plane (generalizes to higher dimensions)
  - Function $f : D \rightarrow D$
  - There exists some $x$ such that $f(x) = x$



# Intuition

- Consider the one-dimensional case: mapping a line segment onto itself
- $x \in [0, 1]$
- $f(x) \in [0, 1]$
- There must exist some x for which $f(x) = x$
- Examples (in 2D)
  - A mall directory
  - Crumpling up a piece of graph paper






# Back to dataflow

- Game plan:
  - Finite partially ordered set with least element: $D$
  - Function $f : D \rightarrow D$
  - Monotonic function $f : D \rightarrow D$
  - $\exists$ fixpoint of $f$
    - $\exists$ *least* fixpoint of $f$
  - Generalization to case when $D$ has a greatest element, $\top$
    - $\exists$ *greatest* fixpoint of $f$
  - Generalization to systems of equations



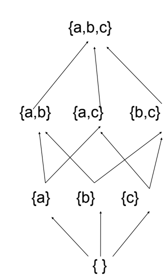
# Partially ordered set (poset)

- Set *D* with a relation $\sqsubseteq$ that is
  - Reflexive: $x \sqsubseteq x$
  - Anti-symmetric: $x \sqsubseteq y$ and $y \sqsubseteq x \Rightarrow y = x$
  - Transitive: $x \sqsubseteq y, y \sqsubseteq z \Rightarrow x \sqsubseteq z$
- Example: set of integers and $\leq$
- Graphical representation of poset
  - Graph in which nodes are elements of *D* and relation $\sqsubseteq$ is indicated by arrows
  - Usually omit reflexive and transitive arrows for legibility
  - Not counting reflexive edges, graph is always a DAG (why?)




# Another example

- Powerset of any set, ordered by $\subseteq$ is a poset
- In the example, poset elements are {}, {a}, {a, b}, {a, b, c}, etc.
- $X \sqsubseteq Y$ iff $X \subseteq Y$



# Finite poset with least element

- Poset in which
  - Set is finite
  - There is a least element that is below all other elements in poset
- Examples
  - Set of integers ordered by ≤ is *not* a finite poset with least element (no least element, not finite)
  - Set of natural numbers ordered by ≤ has a least element (0), but not finite
  - Set of factors of 12, ordered by ≤ has a least element as is finite
  - Powerset example from before is finite (how many elements?) with a least element ({ })



# Domains

- "Finite poset with least element" is a mouthful, so we will abbreviate this to *domain*
- Later, we will add additional conditions to domains that are of interest to us in the context of dataflow analysis
  - (Goal: what is a lattice?)



# Functions on domains

- If $D$ is a domain, we can define a function $f : D \to D$
  - Function maps each element of domain on to another element of the domain
- Example: for $D$ = powerset of {a, b, c}
  - f(x) = x ∪ {a}
  - g(x) = x – {a}
  - h(x) = {a} – x



# Monotonic functions

- A function $f : D \to D$ on a domain $D$ is *monotonic* if
  - $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$
- Note: this is not the same as $x \sqsubseteq f(x)$
  - This means that x is *extensive*
- Intuition: think of f as an electrical circuit mapping input to output
  - If f is monotonic, raising the input voltage raises the output voltage (or keeps it the same)
  - If f is extensive, the output voltage is always the same or more than the input voltage



# Examples

- Domain D is the powerset of {a, b, c}
- Montonic functions:
  - f(x) = { } (why?)
  - f(x) = x ∪ {a}
  - f(x) = x - {a}
- Not monotonic
  - f(x) = {a} - x (why?)
- Extensivity
  - f(x) = x ∪ {a} is monotonic *and* extensive
  - f(x) = x - {a} is monotonic but not extensive
  - f(x) = {a} - x is neither
- What is a function that is extensive, but not monotonic?



# Fixpoints

- Suppose $f : D \to D$.
  - A value $x$ is a *fixpoint* of $f$ if $f(x) = x$
  - $f$ maps $x$ to itself
- Examples: D is a powerset of {a, b, c}
  - Identity function: f(x) = x
    - Every element is a fixpoint
  - f(x) = x ∪ {a}
    - Every set that contains a is a fixpoint
  - f(x) = {a} - x
    - No fixpoints

# Fixpoint theorem

- One form of *Knaster-Tarski Theorem*:

  If $D$ is a domain and $f : D \rightarrow D$ is monotonic, then $f$ has at least one fixpoint

- More interesting consequence:

  If $\perp$ is the least element of $D$, then $f$ has a *least fixpoint*, and that fixpoint is the largest element in the chain

  $\perp$, $f(\perp)$, $f(f(\perp))$, $f(f(f(\perp)))$ ... $f^n(\perp)$

- Least fixpoint: a fixpoint of $f$, x such that, if y is a fixpoint of $f$, then $x \sqsubseteq y$



# Examples

- For domain of powersets, { } is the least element
- For identity function, $f^n(\{\ \})$ is the chain

  { }, { }, { }, ... so least fixpoint is { }, which is correct

- For $f(x) = x \cup \{a\}$, we get the chain

  { }, {a}, {a}, ... so least fixpoint is {a}, which is correct

- For $f(x) = \{a\} - x$, function is not monotonic, so not guaranteed to have a fixpoint!
- Important observation: as soon as the chain repeats, we have found the fixpoint (why?)



# Proof of fixpoint theorem

- First, prove that largest element of chain $f^n(\perp)$ is a fixpoint
- Second, prove that $f^n(\perp)$ is the *least* fixpoint



# Solving equations

- If $D$ is a domain and $f : D \rightarrow D$ is a monotone function on that domain, then the equation $f(x) = x$ has a least fixpoint, given by the largest element in the sequence

  $\perp$, $f(\perp)$, $f(f(\perp))$, $f(f(f(\perp)))$ ...

- Proof follows directly from fixpoint theorem



# Adding a top

- Now let us consider domains with an element $\top$, such that for every point x in the domain, $x \sqsubseteq \top$
- New theorem: if $D$ is a domain with a greatest element $\top$ and $f : D \rightarrow D$ is monotonic, then the equation $x = f(x)$ has a *greatest* solution, and that solution is the smallest element in the sequence

  $\top$, $f(\top)$, $f(f(\top))$, ...

- Proof?



# Multi-argument functions

- If $D$ is a domain, a function $f : D{\times}D \rightarrow D$ is monotonic if it is monotonic in each argument when the other is held constant
- Intuition:
  - Electrical circuit has two inputs
  - If you raise either input while holding the other constant, the output either goes up or stays the same

# Fixpoints of multi-arg functions

- Can generalize fixpoint theorem in a straightforward way
- If *D* is a domain and *f*, *g* : *D*×*D* → *D* are monotonic, the following system of equations has a least fixpoint solution, calculated in the obvious way

  $x = f(x, y)$ and $y = g(x, y)$
- Can generalize this to more than two variables and domains with greatest elements easily



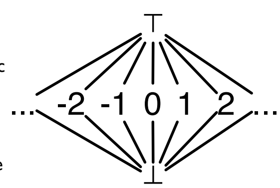
# Lattices

- A bounded *lattice* is a partially ordered set with a ⊥ and ⊤, with two special functions for any pair of points *x* and *y* in the lattice:
  - A *join*: $x \sqcup y$ is the least element that is greater than x and y (also called the *least upper bound*)
  - A *meet*: $x \sqcap y$ is the greatest element that is less than x and y (also called the *greatest lower bound*)
- Are ⊔ and ⊓ monotonic?



# More about lattices

- Bounded lattices with a finite number of elements are a special case of domains with ⊤ (why are they not the same?)
  - Systems of monotonic functions (including ⊔ and ⊓) will have fixpoints
- But some lattices are infinite! (example: the lattice for constant propagation)
  - It turns out that you can show a monotonic function will have a least fixpoint for any lattice (or domain) of *finite height*
  - Finite height: any totally ordered subset of domain (this is called a *chain*) must be finite
  - Why does this work?






# Solving system of equations

- Consider

  $x = f(x, y, z)$

  $y = g(x, y, z)$

  $z = h(x, y, z)$
- Obvious iterative solution: evaluate every function at every step:

  ⊥ $f(\perp,\perp,\perp)$ ...

  ⊥ $g(\perp,\perp,\perp)$ ...

  ⊥ $h(\perp,\perp,\perp)$ ...



# Worklist algorithm

- Obvious point: only necessary to re-evaluate functions whose "important" inputs have changed
- Worklist algorithm
  - Initialize worklist with all equations
  - Initialize solution vector S to all ⊥
  - While worklist not empty
    - Get equation from worklist
    - Re-evaluate equation based on S, update entry corresponding to lhs in S
    - Put all equations which use this lhs on their rhs in the worklist
- Claim: the worklist algorithm for constant propagation is an instance of this approach



# Mapping worklist algorithm

- Careful: the "variables" in constant propagation are not the individual variable values in a state vector. Each variable (from a fixpoint perspective) is an entire state vector – there are as many variables as there are edges in the CFG
- Functions:
  - Program statements: $eval(e, V_{in})$
    - These are called *transfer functions*
    - Need to make sure this is monotonic
  - Branches
    - Propagates input state vector to output – trivially monotonic
  - Merges
    - Use join or meet to combine multiple input variables – monotonic by definition

# Constant propagation

- Step 1: choose lattice
  - Use constant lattice (infinite, but finite height)
- Step 2: choose direction of dataflow
  - Run forward through program
- Step 3: create monotonic transfer functions
  - If input goes from $\perp$ to constant, output can only go up. If input goes from constant to $\top$, output goes to $\top$
- Step 4: choose *confluence operator*
  - What do do at merges? For constant propagation, use join