Dataflow Analysis
Program optimizations

• So far we have talked about different kinds of optimizations
  • Peephole optimizations
  • Local common sub-expression elimination
  • Loop optimizations

• What about global optimizations
  • Optimizations across multiple basic blocks (usually a whole procedure)
    • Not just a single loop
Useful optimizations

- Common subexpression elimination (global)
  - Need to know which expressions are available at a point
- Dead code elimination
  - Need to know if the effects of a piece of code are never needed, or if code cannot be reached
- Constant folding
  - Need to know if variable has a constant value
- Loop invariant code motion
  - Need to know where and when variables are live
- So how do we get this information?
Dataflow analysis

• Framework for doing compiler analyses to drive optimization
• Works across basic blocks
• Examples
  • Constant propagation: determine which variables are constant
  • Liveness analysis: determine which variables are live
  • Available expressions: determine which expressions are have valid computed values
  • Reaching definitions: determine which definitions could “reach” a use
Example: constant propagation

- Goal: determine when variables take on constant values
- Why? Can enable many optimizations
  - Constant folding
    ```
    x = 1;
y = x + 2;
if (x > z) then y = 5
... y ...
    ```
  - Create dead code
    ```
    x = 1;
y = x + 2;
if (y > x) then y = 5
... y ...
    ```
Example: constant propagation

• Goal: determine when variables take on constant values

• Why? Can enable many optimizations

• Constant folding

  x = 1;
  y = x + 2;
  if (x > z) then y = 5
  ... y ...

  x = 1;
  y = 3;
  if (x > z) then y = 5
  ... y ...

• Create dead code

  x = 1;
  y = x + 2;
  if (y > x) then y = 5
  ... y ...

  x = 1;
  y = x + 2;
  if (y > x) then y = 5
  ... y ...
Example: constant propagation

- Goal: determine when variables take on constant values
- Why? Can enable many optimizations
  - Constant folding
    
    \[
    \begin{align*}
    x &= 1; \\
    y &= x + 2; \\
    \text{if} \ (x > z) \ \text{then} \ y &= 5 \end{align*}
    \]
    ...
    \[
    \begin{align*}
    x &= 1; \\
    y &= 3; \\
    \text{if} \ (x > z) \ \text{then} \ y &= 5 \end{align*}
    \]
    ...

  - Create dead code
    
    \[
    \begin{align*}
    x &= 1; \\
    y &= x + 2; \\
    \text{if} \ (y > x) \ \text{then} \ y &= 5 \end{align*}
    \]
    ...
    \[
    \begin{align*}
    x &= 1; \\
    y &= 3; \ //\text{dead code} \\
    \text{if} \ (\text{true}) \ \text{then} \ y &= 5 \ //\text{simplify!} \end{align*}
    \]
    ...

How can we find constants?

• Ideal: run program and see which variables are constant
  • Problem: variables can be constant with some inputs, not others – need an approach that works for all inputs!
  • Problem: program can run forever (infinite loops?) – need an approach that we know will finish
• Idea: run program symbolically
  • Essentially, keep track of whether a variable is constant or not constant (but nothing else)
Overview of algorithm

- Build control flow graph
  - We’ll use statement-level CFG (with merge nodes) for this
- Perform symbolic evaluation
  - Keep track of whether variables are constant or not
- Replace constant-valued variable uses with their values, try to simplify expressions and control flow
x = 1;
y = x + 2;
if (y > x) then y = 5;
... y ...

Build CFG

start

x = 1

y = x + 2

y > x ?

y = 5

merge

... y ...

end
Symbolic evaluation

- Idea: replace each value with a symbol
- constant (specify which), maybe constant, definitely not constant
- Can organize these possible values in a lattice (will formalize this later)
Symbolic evaluation

• Evaluate expressions symbolically: 
  \( \text{eval}(e, V_{in}) \)

• If \( e \) evaluates to a constant, 
  return that value. If any input is 
  \( \top \) (or \( \bot \)), return \( \top \) (or \( \bot \))

• Why?

• Two special operations on lattice
  • \( \text{meet}(a, b) \) – highest value less 
    than or equal to both \( a \) and \( b \)
  • \( \text{join}(a, b) \) – lowest value greater 
    than or equal to both \( a \) and \( b \)

Join often written as \( a \sqcup b \)
Meet often written as \( a \sqcap b \)
Putting it together

• Keep track of the symbolic value of a variable at every program point (on every CFG edge)
  • State vector
• What should our initial value be?
  • Starting state vector is all \( \top \)
  • Can’t make any assumptions about inputs – must assume not constant
  • Everything else starts as \( \bot \), since we don’t know if the variable is constant or not at that point
Executing symbolically

- For each statement $t = e$
  evaluate $e$ using $V_{in}$, update value for $t$ and propagate state vector to next statement

- What about switches?
  - If $e$ is true or false, propagate $V_{in}$ to appropriate branch

- What if we can’t tell?
  - Propagate $V_{in}$ to both branches, and symbolically execute both sides

- What do we do at merges?
Handling merges

- Have two different $V_{in}$s coming from two different paths

- Goal: want new value for $V_{in}$ to be safe (shouldn’t generate wrong information), and we don’t know which path we actually took

- Consider a single variable. Several situations:
  - $V_1 = \bot, V_2 = * \rightarrow V_{out} = *$
  - $V_1 = \text{constant } x, V_2 = x \rightarrow V_{out} = x$
  - $V_1 = \text{constant } x, V_2 = \text{constant } y \rightarrow V_{out} = T$
  - $V_1 = T, V_2 = * \rightarrow V_{out} = T$

- Generalization:
  - $V_{out} = V_1 \sqcup V_2$
Result: worklist algorithm

- Associate state vector with each edge of CFG, initialize all values to \( \bot \), worklist has just start edge

- While worklist not empty, do:

  Process the next edge from worklist
  Symbolically evaluate target node of edge using input state vector
  If target node is assignment \((x = e)\), propagate \( V_{in}[eval(e)/x] \) to output edge
  If target node is branch \((e?)\)
    If \( eval(e) \) is true or false, propagate \( V_{in} \) to appropriate output edge
    Else, propagate \( V_{in} \) along both output edges
  If target node is merge, propagate \( join(all V_{in}) \) to output edge
  If any output edge state vector has changed, add it to worklist
Running example

start

x = 1

y = x + 2

y > x ?

y = 5

merge

... y ...

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Running example

\begin{itemize}
\item \texttt{start}
\item \texttt{x = 1}
\item \texttt{y = x + 2}
\item \texttt{y > x ?}
\item \texttt{merge}
\item \texttt{... y ...}
\item \texttt{end}
\end{itemize}
What do we do about loops?

• Unless a loop never executes, symbolic execution looks like it will keep going around to the same nodes over and over again

• Insight: if the input state vector(s) for a node don’t change, then its output doesn’t change

• If input stops changing, then we are done!

• Claim: input will eventually stop changing. Why?
Loop example

First time through loop, $x = 1$
Subsequent times, $x = T$
Complexity of algorithm

• $V = \# \text{ of variables, } E = \# \text{ of edges}$

• Height of lattice $= 2 \rightarrow$ each state vector can be updated at most $2 \times V$ times.

• So each edge is processed at most $2 \times V$ times, so we process at most $2 \times E \times V$ elements in the worklist.

• Cost to process a node: $O(V)$

• Overall, algorithm takes $O(EV^2)$ time
Question

- Can we generalize this algorithm and use it for more analyses?
- First, let’s lay the theoretical foundation for dataflow analysis.
Lattice Theory
First, something interesting

• **Brouwer Fixpoint Theorem**
  - Every continuous function $f$ from a closed disk into itself has at least one fixed point

• More formally:
  - Domain $D$: a convex, closed, bounded subspace in a plane (generalizes to higher dimensions)
  - Function $f : D \rightarrow D$
  - There exists some $x$ such that $f(x) = x$
• Consider the one-dimensional case: mapping a line segment onto itself

• \( x \in [0, 1] \)

• \( f(x) \in [0, 1] \)

• There must exist some \( x \) for which \( f(x) = x \)

• Examples (in 2D)
  • A mall directory
  • Crumpling up a piece of graph paper

Intuition
Back to dataflow

- Game plan:
  - Finite partially ordered set with least element: $D$
  - Function $f : D \rightarrow D$
  - Monotonic function $f : D \rightarrow D$
  - $\exists$ fixpoint of $f$
    - $\exists$ least fixpoint of $f$
  - Generalization to case when $D$ has a greatest element, $\top$
    - $\exists$ greatest fixpoint of $f$
  - Generalization to systems of equations
Partially ordered set (poset)

- Set $D$ with a relation $\sqsubseteq$ that is
  - Reflexive: $x \sqsubseteq x$
  - Anti-symmetric: $x \sqsubseteq y$ and $y \sqsubseteq x \Rightarrow y = x$
  - Transitive: $x \sqsubseteq y, y \sqsubseteq z \Rightarrow x \sqsubseteq z$

- Example: set of integers and $\leq$

- Graphical representation of poset
  - Graph in which nodes are elements of $D$ and relation $\sqsubseteq$ is indicated by arrows
  - Usually omit reflexive and transitive arrows for legibility
  - Not counting reflexive edges, graph is always a DAG (why?)
Another example

- Powerset of any set, ordered by $\subseteq$ is a poset
- In the example, poset elements are $\emptyset, \{a\}, \{a, b\}, \{a, b, c\}$, etc.
- $X \subseteq Y$ iff $X \subseteq Y$
Finite poset with least element

- Poset in which
  - Set is finite
  - There is a least element that is below all other elements in poset

- Examples
  - Set of integers ordered by $\leq$ is not a finite poset with least element (no least element, not finite)
  - Set of natural numbers ordered by $\leq$ has a least element (0), but not finite
  - Set of factors of 12, ordered by $\leq$ has a least element as is finite
  - Powerset example from before is finite (how many elements?) with a least element (\{\})
Domains

• “Finite poset with least element” is a mouthful, so we will abbreviate this to *domain*

• Later, we will add additional conditions to domains that are of interest to us in the context of dataflow analysis

• (Goal: what is a lattice?)
Functions on domains

- If $D$ is a domain, we can define a function $f : D \rightarrow D$
  - Function maps each element of domain on to another element of the domain
- Example: for $D =$ powerset of $\{a, b, c\}$
  - $f(x) = x \cup \{a\}$
  - $g(x) = x - \{a\}$
  - $h(x) = \{a\} - x$
Monotonic functions

- A function $f: D \rightarrow D$ on a domain $D$ is **monotonic** if
  - $x \subseteq y \Rightarrow f(x) \subseteq f(y)$

- Note: this is not the same as $x \subseteq f(x)$
- This means that $x$ is **extensive**

- Intuition: think of $f$ as an electrical circuit mapping input to output
  - If $f$ is monotonic, raising the input voltage raises the output voltage (or keeps it the same)
  - If $f$ is extensive, the output voltage is always the same or more than the input voltage
Examples

- Domain D is the powerset of \{a, b, c\}
- Monotonic functions:
  - \( f(x) = \{\} \) (why?)
  - \( f(x) = x \cup \{a\} \)
  - \( f(x) = x - \{a\} \)
- Not monotonic
  - \( f(x) = \{a\} - x \) (why?)
- Extensivity
  - \( f(x) = x \cup \{a\} \) is monotonic and extensive
  - \( f(x) = x - \{a\} \) is monotonic but not extensive
  - \( f(x) = \{a\} - x \) is neither
- What is a function that is extensive, but not monotonic?
Fixpoints

- Suppose $f : D \rightarrow D$.
  - A value $x$ is a fixpoint of $f$ if $f(x) = x$
  - $f$ maps $x$ to itself

- Examples: $D$ is a powerset of $\{a, b, c\}$
  - Identity function: $f(x) = x$
    - Every element is a fixpoint
  - $f(x) = x \cup \{a\}$
    - Every set that contains $a$ is a fixpoint
  - $f(x) = \{a\} - x$
    - No fixpoints
Fixpoint theorem

• One form of **Knaster-Tarski Theorem**:

  If \( D \) is a domain and \( f : D \rightarrow D \) is monotonic, then \( f \) has at least one fixpoint

• More interesting consequence:

  If \( \perp \) is the least element of \( D \), then \( f \) has a **least fixpoint**, and that fixpoint is the largest element in the chain

  \[ \perp, f(\perp), f(f(\perp)), f(f(f(\perp))), ... f^n(\perp) \]

• Least fixpoint: a fixpoint of \( f \), \( x \) such that, if \( y \) is a fixpoint of \( f \), then \( x \sqsubseteq y \)
Examples

• For domain of powersets, \{ \} is the least element
• For identity function, \( f^n(\{ \}) \) is the chain
  \{ \}, \{ \}, \{ \}, \ldots \) so least fixpoint is \{ \}, which is correct
• For \( f(x) = x \cup \{a\} \), we get the chain
  \{ \}, \{a\}, \{a\}, \ldots \) so least fixpoint is \{a\}, which is correct
• For \( f(x) = \{a\} – x \), function is not monotonic, so not guaranteed to have a fixpoint!
• Important observation: as soon as the chain repeats, we have found the fixpoint (why?)
Proof of fixpoint theorem

- First, prove that largest element of chain $f^n(\bot)$ is a fixpoint

- Second, prove that $f^n(\bot)$ is the least fixpoint
Solving equations

• If $D$ is a domain and $f : D \rightarrow D$ is a monotone function on that domain, then the equation $f(x) = x$ has a least fixpoint, given by the largest element in the sequence

  $\bot, f(\bot), f(f(\bot)), f(f(f(\bot))), \ldots$

• Proof follows directly from fixpoint theorem
Adding a top

• Now let us consider domains with an element $\top$, such that for every point $x$ in the domain, $x \sqsubseteq \top$

• New theorem: if $D$ is a domain with a greatest element $\top$ and $f : D \to D$ is monotonic, then the equation $x = f(x)$ has a greatest solution, and that solution is the smallest element in the sequence

\[ \top, f(\top), f(f(\top)), \ldots \]

• Proof?
Multi-argument functions

- If $D$ is a domain, a function $f : D \times D \rightarrow D$ is monotonic if it is monotonic in each argument when the other is held constant.

- Intuition:
  - Electrical circuit has two inputs.
  - If you raise either input while holding the other constant, the output either goes up or stays the same.
Fixpoints of multi-arg functions

• Can generalize fixpoint theorem in a straightforward way

• If $D$ is a domain and $f, g : D \times D \to D$ are monotonic, the following system of equations has a least fixpoint solution, calculated in the obvious way

$$x = f(x, y) \text{ and } y = g(x, y)$$

• Can generalize this to more than two variables and domains with greatest elements easily
Lattices

- A bounded *lattice* is a partially ordered set with a $\perp$ and $\top$, with two special functions for any pair of points $x$ and $y$ in the lattice:
  - A *join*: $x \sqcup y$ is the least element that is greater than $x$ and $y$ (also called the *least upper bound*)
  - A *meet*: $x \sqcap y$ is the greatest element that is less than $x$ and $y$ (also called the *greatest lower bound*)
- Are $\sqcup$ and $\sqcap$ monotonic?
More about lattices

- Bounded lattices with a finite number of elements are a special case of domains with $\top$ (why are they not the same?)
- Systems of monotonic functions (including $\sqcup$ and $\sqcap$) will have fixpoints
- But some lattices are infinite! (example: the lattice for constant propagation)
- It turns out that you can show a monotonic function will have a least fixpoint for any lattice (or domain) of finite height
- Finite height: any totally ordered subset of domain (this is called a chain) must be finite
- Why does this work?
Solving system of equations

• Consider

\[ x = f(x, y, z) \]
\[ y = g(x, y, z) \]
\[ z = h(x, y, z) \]

• Obvious iterative solution: evaluate every function at every step:

\[ \perp \quad f(\perp, \perp, \perp) \quad \ldots \]
\[ \perp \quad g(\perp, \perp, \perp) \quad \ldots \]
\[ \perp \quad h(\perp, \perp, \perp) \quad \ldots \]
Worklist algorithm

- Obvious point: only necessary to re-evaluate functions whose “important” inputs have changed

- Worklist algorithm
  - Initialize worklist with all equations
  - Initialize solution vector $S$ to all ⊥
  - While worklist not empty
    - Get equation from worklist
    - Re-evaluate equation based on $S$, update entry corresponding to lhs in $S$
    - Put all equations which use this lhs on their rhs in the worklist
  - Claim: the worklist algorithm for constant propagation is an instance of this approach
Mapping worklist algorithm

- Careful: the “variables” in constant propagation are not the individual variable values in a state vector. Each variable (from a fixpoint perspective) is an entire state vector – there are as many variables as there are edges in the CFG.

- Functions:
  - Program statements: eval(e, V_{in})
    - These are called *transfer functions*
  - Need to make sure this is monotonic

- Branches
  - Propagates input state vector to output – trivially monotonic

- Merges
  - Use join or meet to combine multiple input variables – monotonic by definition
Constant propagation

• Step 1: choose lattice
  • Use constant lattice (infinite, but finite height)

• Step 2: choose direction of dataflow
  • Run forward through program

• Step 3: create monotonic transfer functions
  • If input goes from $\bot$ to constant, output can only go up. If input goes from constant to $\top$, output goes to $\top$

• Step 4: choose confluence operator
  • What do do at merges? For constant propagation, use join