Dependence Analysis
Motivating question

- Can the loops on the right be run in parallel?
  - i.e., can different processors run different iterations in parallel?
- What needs to be true for a loop to be parallelizable?
  - Iterations cannot interfere with each other
- No dependence between iterations

```c
for (i = 1; i < N; i++) {
    a[i] = b[i];
    c[i] = a[i - 1];
}
```

```c
for (i = 1; i < N; i++) {
    a[i] = b[i];
    c[i] = a[i] + b[i - 1];
}
```
Dependences

- A flow dependence occurs when one iteration writes a location that a later iteration reads.

```c
for (i = 1; i < N; i++) {
    a[i] = b[i];
    c[i] = a[i - 1];
}
```

**Diagram:**

```
i = 1       i = 2       i = 3       i = 4       i = 5
W(a[1])   W(a[2])   W(a[3])   W(a[4])   W(a[5])
R(b[1])   R(b[2])   R(b[3])   R(b[4])   R(b[5])
W(c[1])   W(c[2])   W(c[3])   W(c[4])   W(c[5])
R(a[0])   R(a[1])   R(a[2])   R(a[3])   R(a[4])
```
Running a loop in parallel

- If there is a dependence in a loop, we cannot guarantee that the loop will run correctly in parallel
- What if the iterations run out of order?
  - Might read from a location before the correct value was written to it
- What if the iterations do not run in lock-step?
  - Same problem!
Other kinds of dependence

• **Anti dependence** – When an iteration *reads* a location that a later iteration *writes* (why is this a problem?)

```c
for (i = 1; i < N; i++) {
    a[i - 1] = b[i];
    c[i] = a[i];
}
```

• **Output dependence** – When an iteration *writes* a location that a later iteration *writes* (why is this a problem?)

```c
for (i = 1; i < N; i++) {
    a[i] = b[i];
    a[i + 1] = c[i];
}
```
Data dependence concepts

- Dependence source is the earlier statement (the statement at the tail of the dependence arrow)
- Dependence sink is the later statement (the statement at the head of the dependence arrow)

\[
\begin{align*}
\text{i = 1} & \quad \text{i = 2} & \quad \text{i = 3} & \quad \text{i = 4} & \quad \text{i = 5} \\
W(a[1]) & \quad W(a[2]) & \quad W(a[3]) & \quad W(a[4]) & \quad W(a[5]) \\
R(b[1]) & \quad R(b[2]) & \quad R(b[3]) & \quad R(b[4]) & \quad R(b[5]) \\
W(c[1]) & \quad W(c[2]) & \quad W(c[3]) & \quad W(c[4]) & \quad W(c[5]) \\
R(a[0]) & \quad R(a[1]) & \quad R(a[2]) & \quad R(a[3]) & \quad R(a[4])
\end{align*}
\]

- Dependences can only go forward in time: always from an earlier iteration to a later iteration.
Using dependences

- If there are no dependences, we can parallelize a loop
  - None of the iterations interfere with each other
- Can also use dependence information to drive other optimizations
  - Loop interchange
  - Loop fusion
  - (We will discuss these later)
- Two questions:
  - How do we represent dependences in loops?
  - How do we determine if there are dependences?
Representing dependences

• Focus on flow dependences for now

• Dependences in straight line code are easy to represent:
  • One statement writes a location (variable, array location, etc.) and another reads that same location
  • Can figure this out using reaching definitions

• What do we do about loops?

  • We often care about dependences between the same statement in different iterations of the loop!

    ```
    for (i = 1; i < N; i++) {
      a[i + 1] = a[i] + 2
    }
    ```
Iteration space graphs

• Represent each *dynamic* instance of a loop as a point in a graph

• Draw arrows from one point to another to represent dependences

```plaintext
for (i = 0; i < N; i++) {
    a[i + 2] = a[i]
}
```
Iteration space graphs

• Represent each *dynamic* instance of a loop as a point in a graph

• Draw arrows from one point to another to represent dependences

  for (i = 0; i < N; i++) {
    a[i + 2] = a[i]
  }

• Step 1: Create nodes, 1 for each iteration

  • Note: not 1 for each array location!

0  1  2  3  4  5
Iteration space graphs

• Represent each *dynamic* instance of a loop as a point in a graph

• Draw arrows from one point to another to represent dependences

```c
for (i = 0; i < N; i++) {
    a[i + 2] = a[i]
}
```

• Step 2: Determine which array elements are read and written in each iteration

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
</table>
Iteration space graphs

- Represent each *dynamic* instance of a loop as a point in a graph.
- Draw arrows from one point to another to represent dependences.

```c
for (i = 0; i < N; i++) {
  a[i + 2] = a[i]
}
```

- Step 3: Draw arrows to represent dependences.
2-D iteration space graphs

- Can do the same thing for doubly-nested loops
- 2 loop counters

```c
for (i = 0; i < N; i++)
  for (j = 0; j < N; j++)
    a[i+1][j-2] = a[i][j] + 1
```
Iteration space graphs

- Can also represent output and anti dependences
- Use different kinds of arrows for clarity. E.g.
  - for output
  - for anti

- Crucial problem: Iteration space graphs are potentially infinite representations!
- Can we represent dependences in a more compact way?
Distance and direction vectors

• Compiler researchers have devised *compressed* representations of dependences
  • Capture the same dependences as an iteration space graph
  • May lose *precision* (show more dependences than the loop actually has)

• Two types
  • Distance vectors: captures the “shape” of dependences, but not the particular source and sink
  • Direction vectors: captures the “direction” of dependences, but not the particular shape
Distance vector

- Represent each dependence arrow in an iteration space graph as a vector
- Captures the “shape” of the dependence, but loses where the dependence originates

Distance vector for this iteration space: (2)
- Each dependence is 2 iterations forward
2-D distance vectors

• Distance vector for this graph:
  • (1, -2)
  • +1 in the i direction, -2 in the j direction

• Crucial point about distance vectors: they are always “positive”
  • First non-zero entry has to be positive
  • Dependences can’t go backwards in time
More complex example

- Can have multiple distance vectors

```c
for (i = 0; i < N; i++)
    for (j = 0; j < N; j++)
        a[i+1][j-2] = a[i][j] + a[i-1][j-2]
```
More complex example

- Can have multiple distance vectors

```c
for (i = 0; i < N; i++)
    for (j = 0; j < N; j++)
        a[i+1][j-2] = a[i][j] + a[i-1][j-2]
```

- Distance vectors
  - (1, -2)
  - (2, 0)

- Important point: order of vectors depends on order of loops, not use in arrays
Problems with distance vectors

- The preceding examples show how distance vectors can summarize all the dependences in a loop nest using just a small number of distance vectors.
- Can’t always summarize as easily.
- Running example:

  ```java
  for (i = 0; i < N; i++)
    a[2*i] = a[i];
  ```

  ![Diagram with nodes labeled 0 to 6 and arrows showing dependencies between elements in array a.]

  - Write: a[0], a[2], a[4], a[6], a[8], a[10], a[12]
  - Read: a[0], a[2], a[2], a[3], a[4], a[5], a[6]
Loss of precision

- What are the distance vectors for this code?
  - (1), (2), (3), (4) ...
- Note: we have information about the length of each vector, but not about the source of each vector
- What happens if we try to reconstruct the iteration space graph?
Loss of precision

• What are the distance vectors for this code?
  • (1), (2), (3), (4) ...

• Note: we have information about the length of each vector, but not about the source of each vector

• What happens if we try to reconstruct the iteration space graph?
Direction vectors

• The whole point of distance vectors is that we want to be able to succinctly capture the dependences in a loop nest
  • But in the previous example, not only did we add a lot of extra information, we still had an infinite number of distance vectors

• Idea: summarize distance vectors, and save only the direction the dependence was in
  • \((2, -1) \rightarrow (+, -)\)
  • \((0, 1) \rightarrow (0, +)\)
  • \((0, -2) \rightarrow (0, -)\)
    • (can’t happen; dependences have to be positive)
  • Notation: sometimes use ‘<‘ and ‘>’ instead of ‘+’ and ‘−’
Why use direction vectors?

- Direction vectors lose a lot of information, but do capture some useful information
  - Whether there is a dependence (anything other than a ‘0’ means there is a dependence)
  - Which dimension and direction the dependence is in
- Many times, the only information we need to determine if an optimization is legal is captured by direction vectors
  - Loop parallelization
  - Loop interchange
Loop parallelization
Loop-carried dependence

• The key concept for parallelization is the *loop carried dependence*

• A dependence that crosses loop iterations

• If there is a loop carried dependence, then that loop *cannot* be parallelized

• Some iterations of the loop depend on other iterations of the same loop
Examples

for (i = 0; i < N; i++)
    a[2*i] = a[i];

Later iterations of i loop depend on earlier iterations

for (i = 0; i < N; i++)
    for (j = 0; j < N; j++)
        a[i+1][j-2] = a[i][j] + 1

Later iterations of both i and j loops depend on earlier iterations
Some subtleties

- Dependences might only be carried over one loop!

\[
\begin{align*}
\text{for } (i = 0; i < N; i++) \\
\text{for } (j = 0; j < N; j++) \\
a[i][j+1] &= a[i][j] + 1
\end{align*}
\]

- Can parallelize i loop, but not j loop
Some subtleties

- Dependences might only be carried over one loop!

\[
\begin{align*}
&\text{for (}i = 0; i < N; i++\text{)} \\
&\text{for (}j = 0; j < N; j++\text{)} \\
&\quad a[i+1][j] = a[i-1][j] + 1
\end{align*}
\]

- Can parallelize j loop, but not i loop
Direction vectors

- So how do direction vectors help?
  - If there is a non-zero entry for a loop dimension, that means that there is a loop carried dependence over that dimension
  - If an entry is zero, then that loop can be parallelized!
Improving parallelism

- Important point: any dependence can prevent parallelization.
- Anti and output dependences are important, not just flow dependences.
- But anti and output dependences can be removed by using more storage.
  - Like register renaming in out-of-order processors.
- In principle, all anti and output dependences can be removed, but this is difficult.
- Key question: when are there flow dependences?

```plaintext
for (i = 0; i < N; i++)
a[i] = a[i + 1] + 1

for (i = 0; i < N; i++)
aa[i] = a[i + 1] + 1
```
Data Dependence Tests
Problem formulation

• Given the loop nest:

    for (i = 0; i < N; i++)
        a[f(i)] = ...
        ... = a[g(i)]

• A dependence exists if there exist an integer i and an i’ such that:
  • f(i) = g(i’)
  • 0 ≤ i, i’ < N
  • If i < i’, write happens before read (flow dependence)
  • If i > i’, write happens after read (anti dependence)
Loop normalization

- Loops that skip iterations can always be normalized to loops that don’t, so we only need to consider loops that have unit strides.

- Note: this is essentially of the reverse of linear test replacement

```c
for (i = L; i < U; i += S)
  ... a[i] ...
```

```
  for (i = 0; i < (U - L)/S; i += 1)
  ... a[S*i + L] ...
```
Diophantine equations

- An equation whose coefficients and solutions are all integers is called a **Diophantine equation**

- Our question:

\[ f(i) = a_i + b \quad g(i) = c_i + d \]

Does \( f(i) = g(i') \) have a solution?

- \( f(i) = g(i') \Rightarrow a_i + b = c_i' + d \Rightarrow a_i + a_2i' = a_3 \)
Solutions to Diophantine eqns

• An equation $a_1^*i + a_2^*i' = a_3$ has a solution iff $\gcd(a_1, a_2)$ evenly divides $a_3$

• Examples
  • $15^*i + 6^*j - 9^*k = 12$ has a solution ($\gcd = 3$)
  • $2^*i + 7^*j = 3$ has a solution ($\gcd = 1$)
  • $9^*i + 6^*j = 10$ has no solution ($\gcd = 3$)
Why does this work?

- Suppose \( g \) is the gcd(a, b) in \( a*i + b*j = c \)
- Can rewrite equation as
  \[
  g*(a'*i + b'*j) = c
  \]
  \[
  a' * i + b' * j = c/g
  \]
- \( a' \) and \( b' \) are integers, and relatively prime (gcd = 1) so by choosing \( i \) and \( j \) correctly, can produce any integer, but only integers
- Equation has a solution provided \( c/g \) is an integer
Finding the GCD

• Finding GCD with Euclid’s algorithm

• Repeat

  \[ a = a \mod b \]

  swap a and b

  until b is 0 (resulting \( a \) is the gcd)

• Why? If \( g \) divides \( a \) and \( b \), then \( g \) divides \( a \mod b \)

\[
gcd(27, 12): \quad a = 27, \quad b = 15
\]
\[
a = 27 \mod 15 = 12
\]
\[
a = 15 \mod 12 = 3
\]
\[
a = 12 \mod 3 = 0
\]
\[
gcd = 3
\]
Downsides to GCD test

• If \( f(i) = g(i') \) fails the GCD test, then there is no \( i, i' \) that can produce a dependence → loop has no dependences

• If \( f(i) = g(i') \), there might be a dependence, but might not

  • \( i \) and \( i' \) that satisfy equation might fall outside bounds

  • Loop may be parallelizable, but cannot tell

• Unfortunately, most loops have \( \text{gcd}(a, b) = 1 \), which divides everything

• Other optimizations (loop interchange) can tolerate dependences in certain situations
Other dependence tests

• GCD test: doesn’t account for loop bounds, does not provide useful information in many cases

• Banerjee test (Utpal Banerjee): accurate test, takes directions and loop bounds into account

• Omega test (William Pugh): even more accurate test, precise but can be very slow

• Range test (Blume and Eigenmann): works for non-linear subscripts

• Compilers tend to perform simple tests and only perform more complex tests if they cannot determine existence of dependence
Other loop optimizations
Loop interchange

- We’ve seen this one before
- Interchange doubly-nested loop to
  - Improve locality
  - Improve parallelism
    - Move parallel loop to outer loop (coarse grained parallelism)
Loop interchange legality

- We noted that loop interchange is not always legal, because it reorders a computation
- Can we use dependences to determine legality?
Loop interchange dependences

- Consider interchanging the following loop, with the dependence graph to the right:

```plaintext
for (i = 0; i < N; i++)
  for (j = 0; j < N; j++)
    a[i+1][j+2] = a[i][j] + 1
```

- Distance vector (1, 2)
- Direction vector (+, +)
Loop interchange dependences

• Consider interchanging the following loop, with the dependence graph to the right:

```c
for (j = 0; j < N; j++)
    for (i = 0; i < N; i++)
        a[i+1][j+2] = a[i][j] + 1
```

• Distance vector (2, 1)
• Direction vector (+, +)
• Distance vector gets swapped!
Loop interchange legality

• Interchanging two loops swaps the order of their entries in distance/direction vectors

  • \((0, +) \rightarrow (+, 0)\)
  • \((+, 0) \rightarrow (0, +)\)

• But remember, we can’t have backwards dependences

  • \((+, -) \rightarrow (-, +)\)

• Illegal dependence → Loop interchange not legal!
Loop interchange dependences

- Example of illegal interchange:

```c
for (i = 0; i < N; i++)
for (j = 0; j < N; j++)
a[i+1][j-2] = a[i][j] + 1
```
Loop interchange dependences

- Example of illegal interchange:

```c
for (j = 0; j < N; j++)
for (i = 0; i < N; i++)
a[i+1][j-2] = a[i][j] + 1
```

- Flow dependences turned into anti-dependences

- Result of computation will change!
Loop fusion/distribution

• Loop fusion: combining two loops into a single loop
  • Improves locality, parallelism
• Loop distribution: splitting a single loop into two loops
  • Can increase parallelism (turn a non-parallelizable loop into a parallelizable loop)
• Legal as long as optimization maintains dependences
  • Every dependence in the original loop should have a dependence in the optimized loop
  • Optimized loop should not introduce new dependences
Fusion/distribution example

- Code 1:
  
  ```
  for (i = 0; i < N; i++)
      a[i - 1] = b[i]

  for (j = 0; j < N; j++)
      c[j] = a[j]
  ```

- Dependence graph

- All red iterations finish before blue iterations \(\rightarrow\) flow dependence

- Code 2:

  ```
  for (i = 0; i < N; i++)
      a[i - 1] = b[i]

  c[i] = a[i]
  ```

- Dependence graph

- i iterations finish before i + 1 iterations \(\rightarrow\) flow dependence
now an anti dependence!
Fusion/distribution utility

for (i = 0; i < N; i++)
a[i] = a[i - 1]

for (j = 0; j < N; j++)
b[j] = a[j]

Fusion

for (i = 0; i < N; i++)
a[i] = a[i - 1]

b[i] = a[i]

Distribution

Fusion and distribution both legal

Right code has better locality, but cannot be parallelized due to loop carried dependences

Left code has worse locality, but blue loop can be parallelized