REDUCED-RANK ADAPTIVE FILTERING AND APPLICATIONS

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Importance of Reduced-Rank Adaptive Filtering

- *Wiener filter* (WF) estimate of $d_0[n]$ from observation $\mathbf{x}_0[n]$ is *MMSE* optimal, *Bayesian* optimal if $d_0[n]$ and $\mathbf{x}_0[n]$ are jointly Gaussian
- WF is employed in many applications because it is easily implemented and only relies on second order statistics
- However, WF depends upon inverse of covariance matrix, $\mathbf{R}_{\mathbf{x_0}}$
- If $\mathbf{x}_0[n]$ is high dimensional, reduced-rank approach is needed to reduce computational complexity and lessen sample support requirements
- Current strong need for reduced-rank adaptive filtering arises from growing disparity between number of degrees of freedom in 3G/4G wireless systems and limitations on sample support due to high mobility and/or high sensitivity to small perturbations

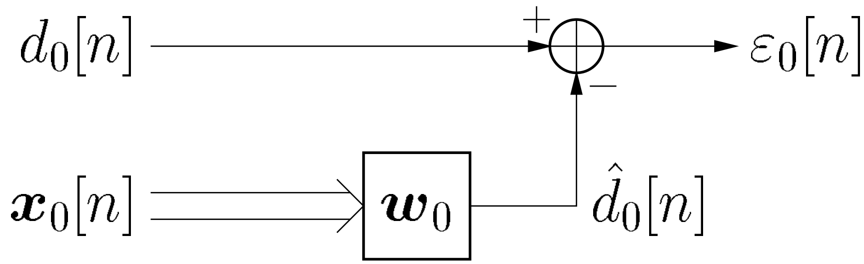
MSNWF vs Principal Components Reduced-Rank Filtering

- *Principal Components* (PC) method: observation vector $\mathbf{x}_0[n]$ transformed to lower dimensionality via a matrix composed of eigenvectors belonging to principal eigenvalues
  - principal eigenvectors need to be estimated and tracked
- PC method only takes into account statistics of $\mathbf{x}_0[n]$
- *Cross-Spectral Metric* (CSM) of Goldstein and Reed: selects those eigenvectors maximizing metric involving $\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}$
  - does not choose the principal eigenvectors, in general
- Goldstein, Reed, & Scharf ultimately presented the *Multi-Stage Nested Wiener Filter* (MSNWF):
  - showed that rank reduction based on eigenvectors is suboptimum
  - MSNWF does not require computation of eigenvectors and is thus computationally advantageous as well

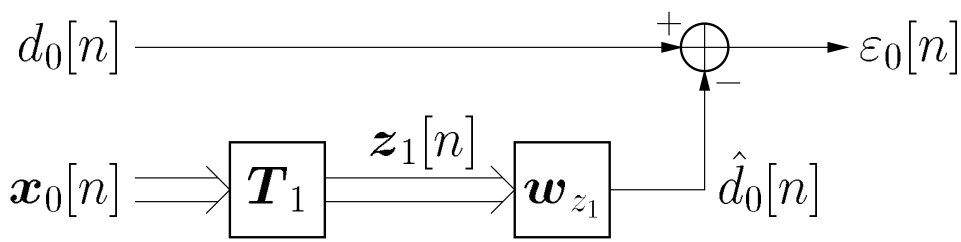
Highly Successful Applications of MSNWF

- Goldstein and Reed have sucessfully applied MSNWF to broad spectrum of radar signal processing problems
- Honig applied MSNWF to Multi-User Access Interference (MAI) suppression for asynchronous CDMA operating in code-space where weight vector dimensionality can be quite high
  - showed number of MSNWF stages needed under heavy loading is mere fraction of subspace dimension required by eigen-space based methods
- Zoltowski has applied the MSNWF to interference suppression for GPS receivers and equalization for the forward-link CDMA with long code
- Willsky has not applied MSNWF per se, but has applied Krylov subspace estimation principles to the problem of error variance estimation in multi-resolution image processing

Full-Rank Wiener Filtering



(A) Weiner Filter



(B) Same but with Full Rank (Square) Matrix Pre-Filtering.

$$\mathbf{T}_1 = \begin{bmatrix} \mathbf{h}_1^H \\ \mathbf{B}_1 \end{bmatrix}, \quad \text{where } \mathbf{B}_1 \text{ is Blocking Matrix: } \mathbf{B}_1\mathbf{h}_1 = \mathbf{0}.$$

Foundational Development of MSNWF

- $\mathbf{h}_1$ is Matched-Filter under AWGN assumption:

$$\mathbf{h}_1 = \frac{\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}}{\|\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}\|_2} \in \mathbf{C}^N.$$

- $\mathbf{B}_1$ is blocking matrix:

$$\mathbf{B}_1\mathbf{h}_1 = \mathbf{0} \qquad \text{or} \quad \mathbf{B}_1 = \text{null}(\mathbf{h}_1^H)^H$$

- Solution to Wiener-Hopf equation associated with transformed system:

$$\mathbf{w}_{\mathbf{z_1}} = \mathbf{R}_{\mathbf{z_1}}^{-1}\mathbf{r}_{\mathbf{z_1},\mathbf{d_0}} \in \mathbf{C}^N, \quad \text{where:} \quad \mathbf{R}_{\mathbf{z_1}} = \begin{bmatrix} \sigma_{d_1}^2 & \mathbf{r}_{\mathbf{x_1},\mathbf{d_1}}^H \\ \mathbf{r}_{\mathbf{x_1},\mathbf{d_1}} & \mathbf{R}_{\mathbf{x_1}} \end{bmatrix} \in \mathbf{C}^{N\times N}$$

- By design, cross-correlation between $\mathbf{z}_1[n]$ and $d_1[n]$ is scalar multiple of standard basis vector $\mathbf{e}_1$
  - $\mathbf{e}_i$ has a one in i-th position: and zeroes elsewhere

$$\mathbf{r}_{\mathbf{z_1},\mathbf{d_0}} = \mathbf{T}_1\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}} = \|\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}\|_2 \;\; \mathbf{e}_1 \in \mathbf{R}^N$$

Foundational Development of MSNWF

- Thus, $\mathbf{w_{z_1}} = \mathbf{R}_{\mathbf{z_1}}^{-1}\mathbf{r}_{\mathbf{z_1},\mathbf{d_0}}$ is first column of $\mathbf{R}_{\mathbf{z_1}}^{-1}$:
- Applying matrix inversion lemma:

$$\mathbf{w_{z_1}} = \alpha_1 \begin{bmatrix} 1 \\ -\mathbf{R}_{\mathbf{x_1}}^{-1}\mathbf{r}_{\mathbf{x_1},\mathbf{d_1}} \end{bmatrix} \in \mathbf{C}^N,$$

- α_1 may be interpreted as scalar WF for estimating $d_0[n]$ from error $\varepsilon_1[n]$

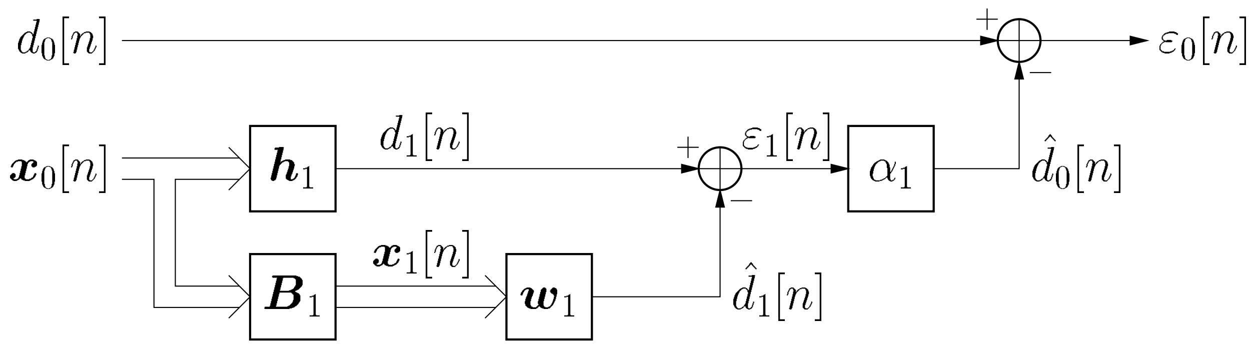
$$\alpha_1 = \|\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}\|_2(\sigma_{d_1}^2 - \mathbf{r}_{\mathbf{x_1},\mathbf{d_1}}^H \mathbf{R}_{\mathbf{x_1}}^{-1}\mathbf{r}_{\mathbf{x_1},\mathbf{d_1}})^{-1}$$

- Scalar Wiener-Hopf equation is $\boxed{\mathcal{E}\{|\varepsilon_1[n]|^2\}\alpha_1 = \mathcal{E}\{\varepsilon_1^*[n]d_0[n]\}}$, where:

$$\mathcal{E}\{|\varepsilon_1[n]|^2\} = \sigma_{d_1}^2 - \mathbf{r}_{\mathbf{x_1},\mathbf{d_1}}^H \mathbf{w}_1 = \sigma_{d_1}^2 - \mathbf{r}_{\mathbf{x_1},\mathbf{d_1}}^H \mathbf{R}_{\mathbf{x_1}}^{-1}\mathbf{r}_{\mathbf{x_1},\mathbf{d_1}}$$

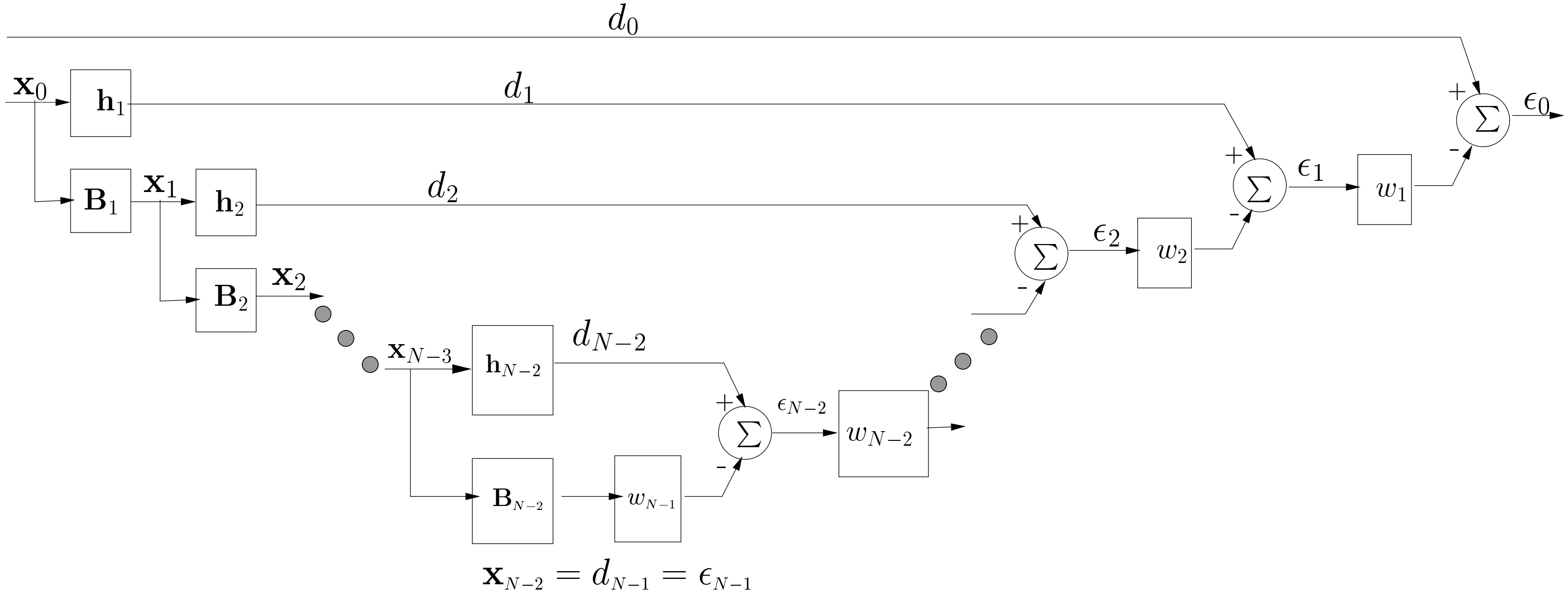
$$\mathcal{E}\{\varepsilon_1^*[n]d_0[n]\} = \mathbf{h}_1^H \mathbf{r}_{\mathbf{x_1},\mathbf{d_1}} = \|\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}\|_2$$

MSNWF After First Stage



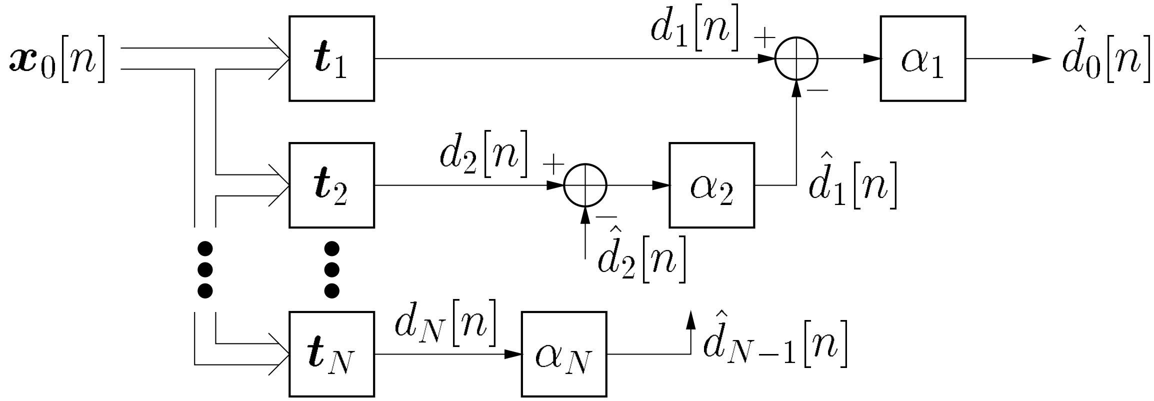
- second stage of decomposition: output of WF $\mathbf{w}_1$ with dimension $N-1$ replaced by weighted error signal $\varepsilon_2[n]$ of WF that estimates output $d_2[n]$ of MF $\mathbf{h}_2$ from blocking-matrix output $\mathbf{x}_2[n] = \mathbf{B}_2\mathbf{x}_1[n]$
- Following this through N stages yields original formulation of MSNWF

Original Structure of MSNWF



- reduced-rank MSNWF obtained by stopping decomposition after $D-1$ steps and replacing last Wiener filter $\mathbf{w}_{D-1}$ by matched filter $\mathbf{h}_{D-1}$
- *Note 1*: correlation matrix of $d_i[n], i = 1, \ldots, N$, is tri-diagonal
- *Note 2*: correlation matrix of $\varepsilon_i[n], i = 1, \ldots, N$, is diagonal $\Rightarrow$ errors are uncorrelated

Filter Bank Implementation of MSNWF

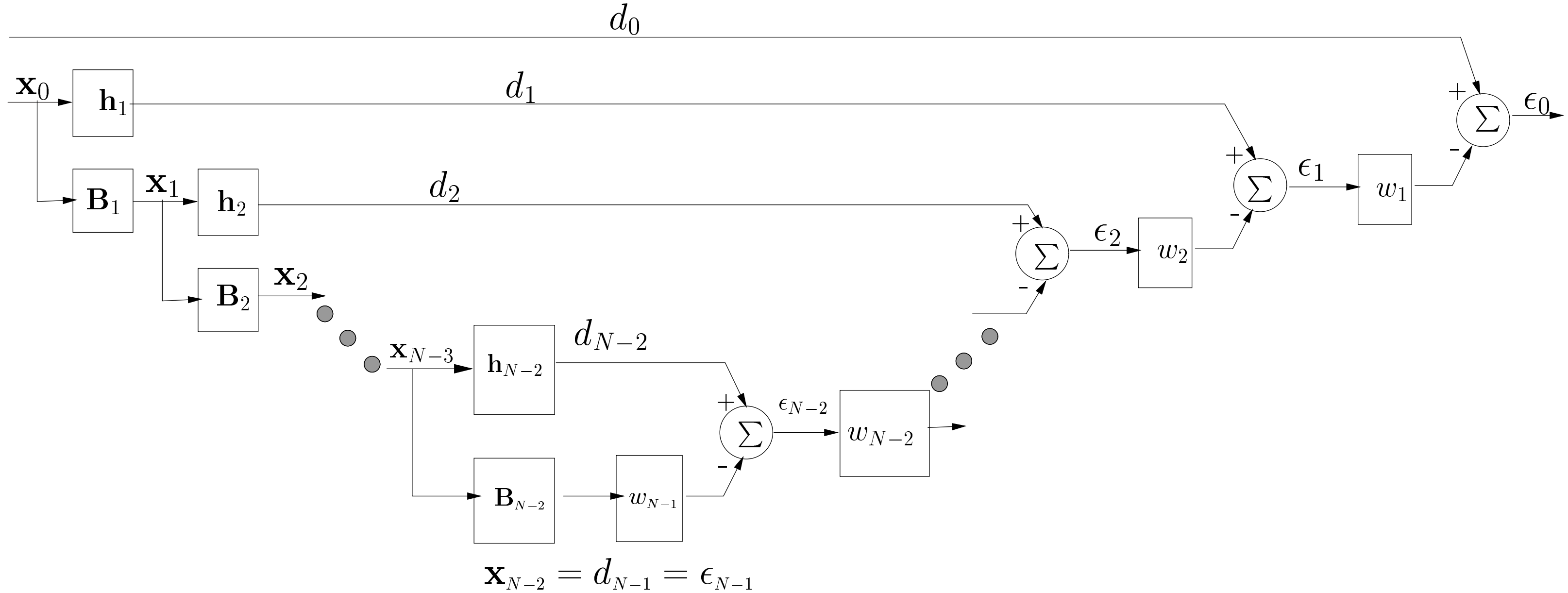


- each "desired" signal $d_i[n], i = 1, \ldots, N$, is output of length N filter

$$\mathbf{t}_i = \left(\prod_{k=1}^{i-1} \mathbf{B}_k^H \right) \mathbf{h}_i \in \mathbf{C}^N.$$

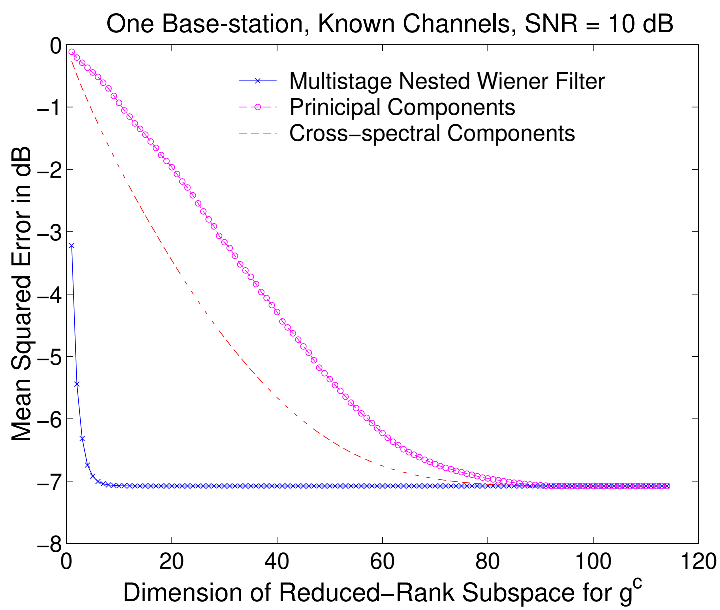
- don't need to form covariance matrix $\mathbf{R}_{\mathbf{x_0}}$
- at i-th stage, WF is replaced by normalized matched filter equal to cross-correlation between $\mathbf{x}_i[n]$ and $d_i[n]$

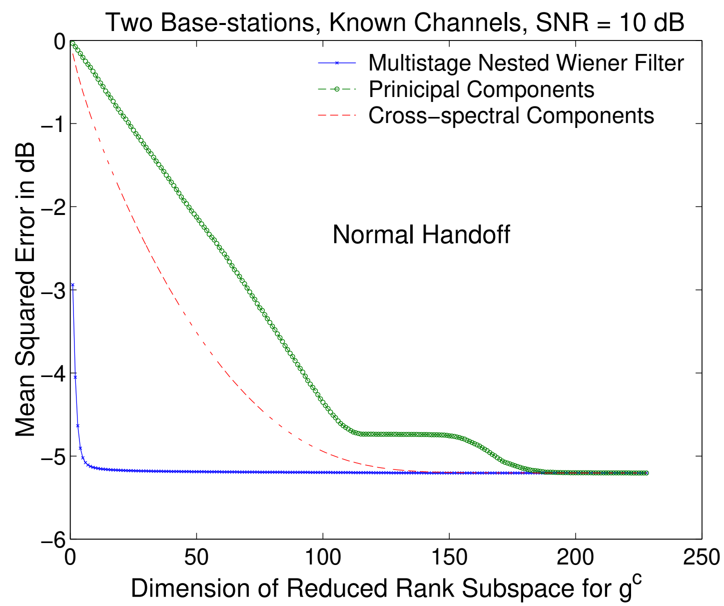
Drawbacks of Original MSNWF Structure



- requires a *forward recursion* to determine $\mathbf{h}_i$'s
- *backwards recursion* then executed to determine scalar WF w_i's
- *Drawback 1*: computational burden of forming blocking matrices
- *Drawback 2*: scalar Weiner weights w_i change completely each time a new stage is added

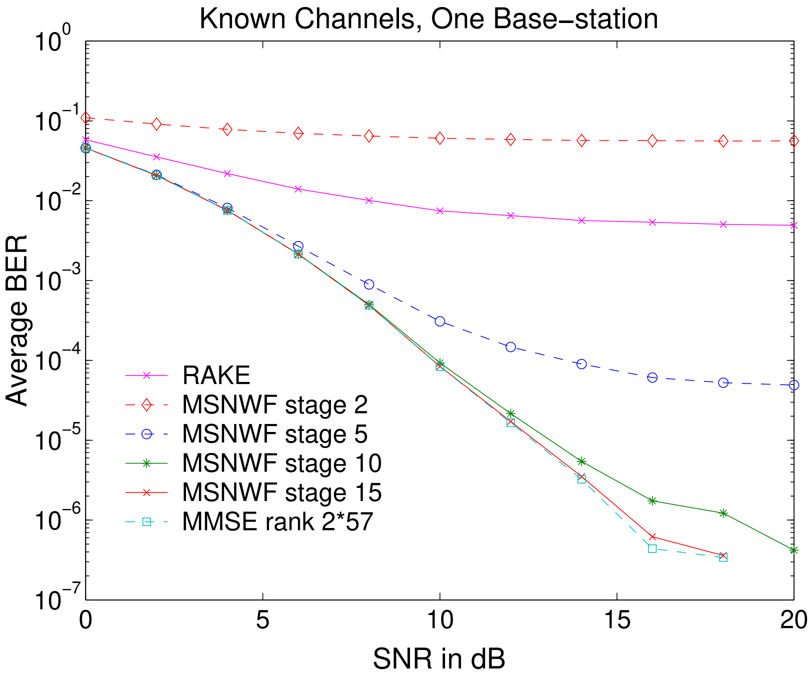
Example: Forward Link Equalization for 3G CDMA

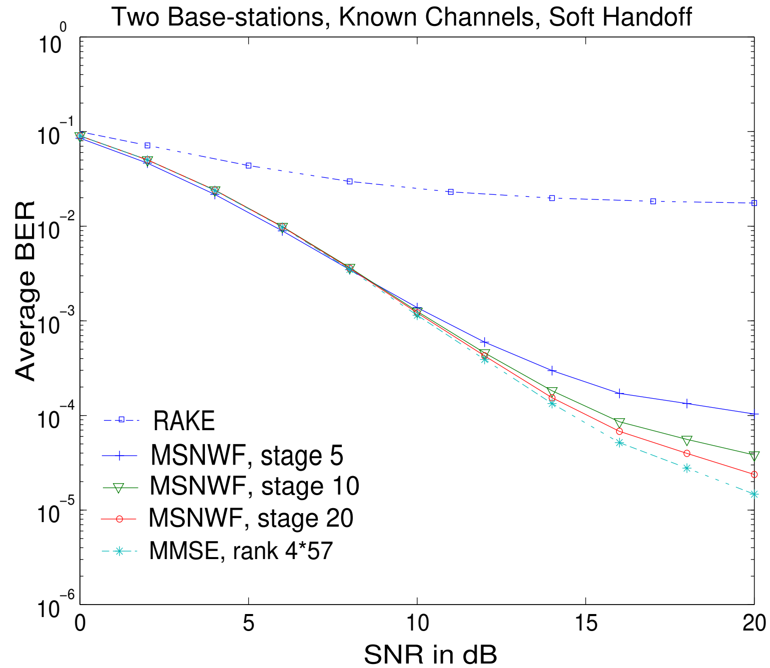




MSE vs Rank of Reduced Dimension Subspace.

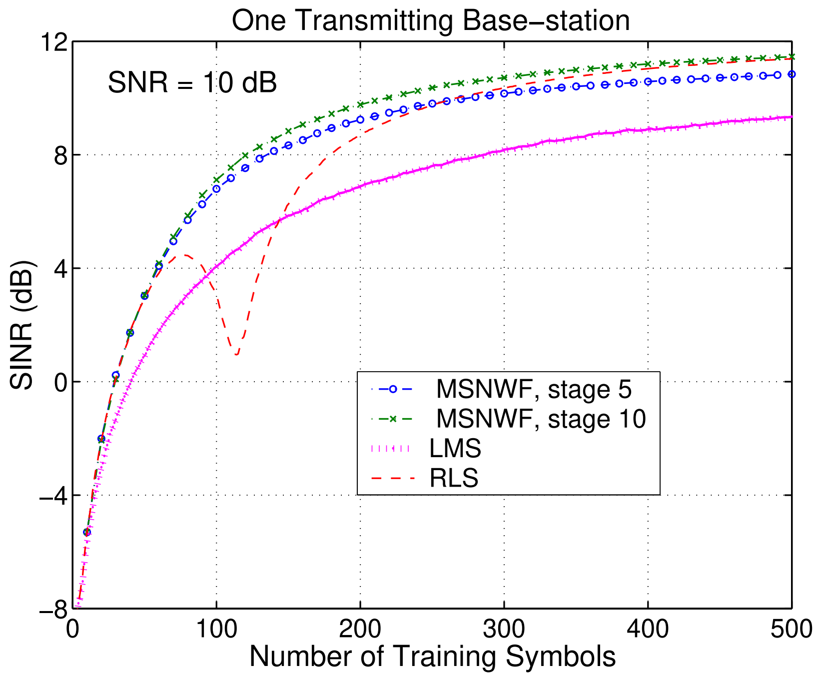
Example: Forward Link Equalization for 3G CDMA

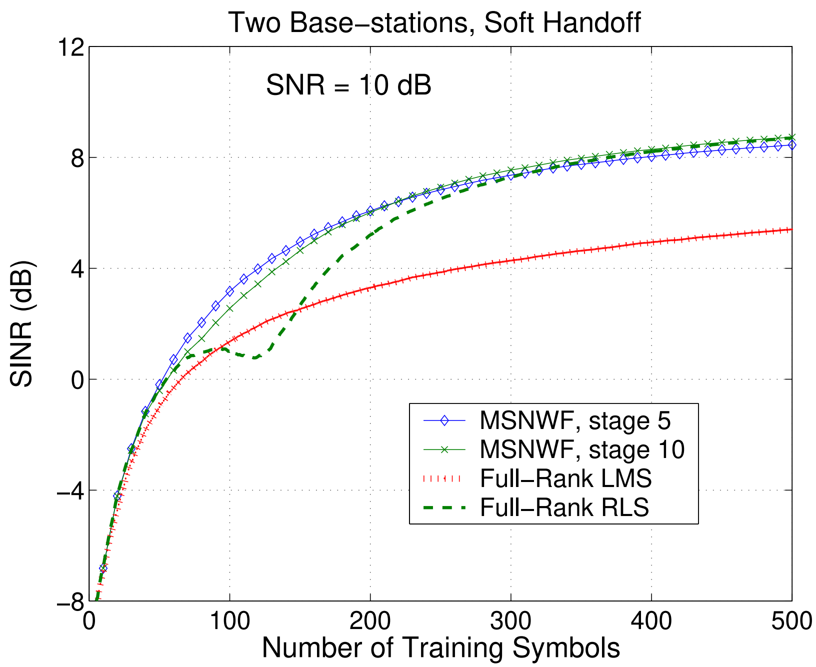




BER for Different Chip-level Equalizers for CDMA Downlink.

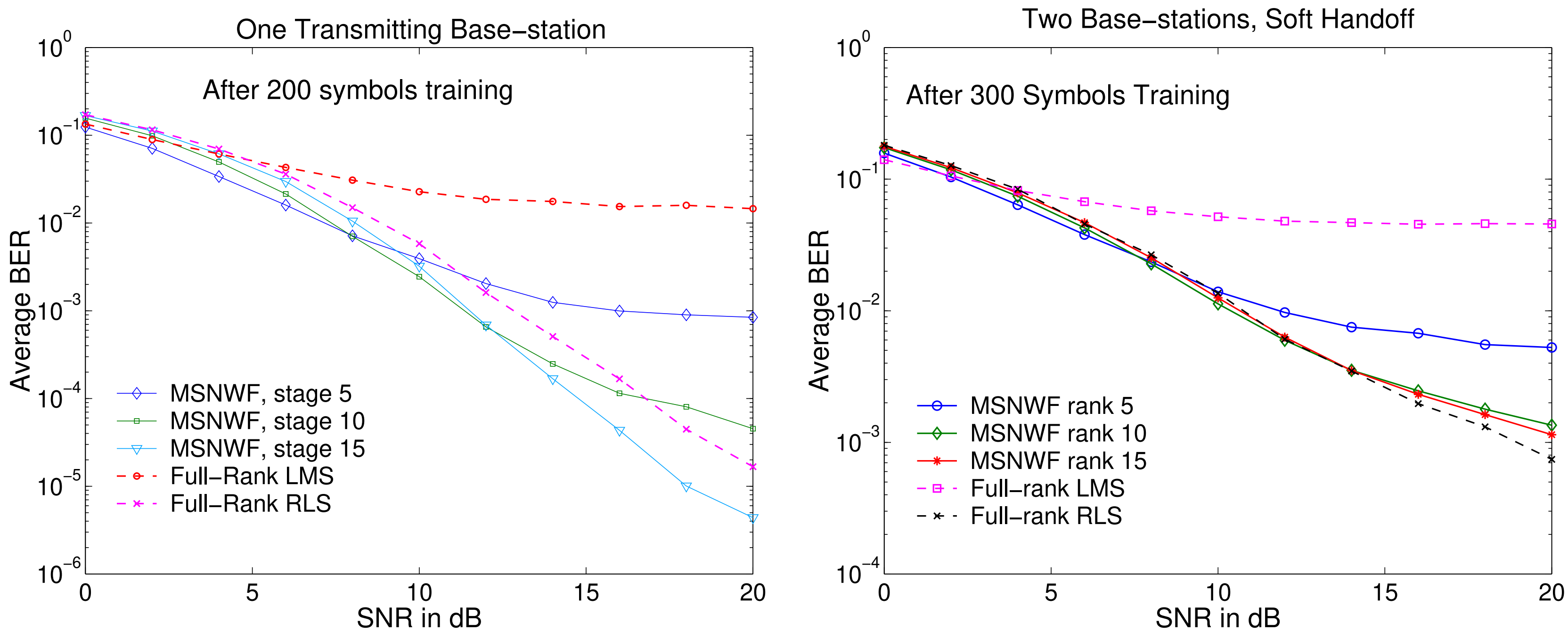
Example: Forward Link Equalization for 3G CDMA





Output SINR vs Time for Adaptive Chip-level Equalizers for CDMA Downlink.

Example: Forward Link Equalization for 3G CDMA



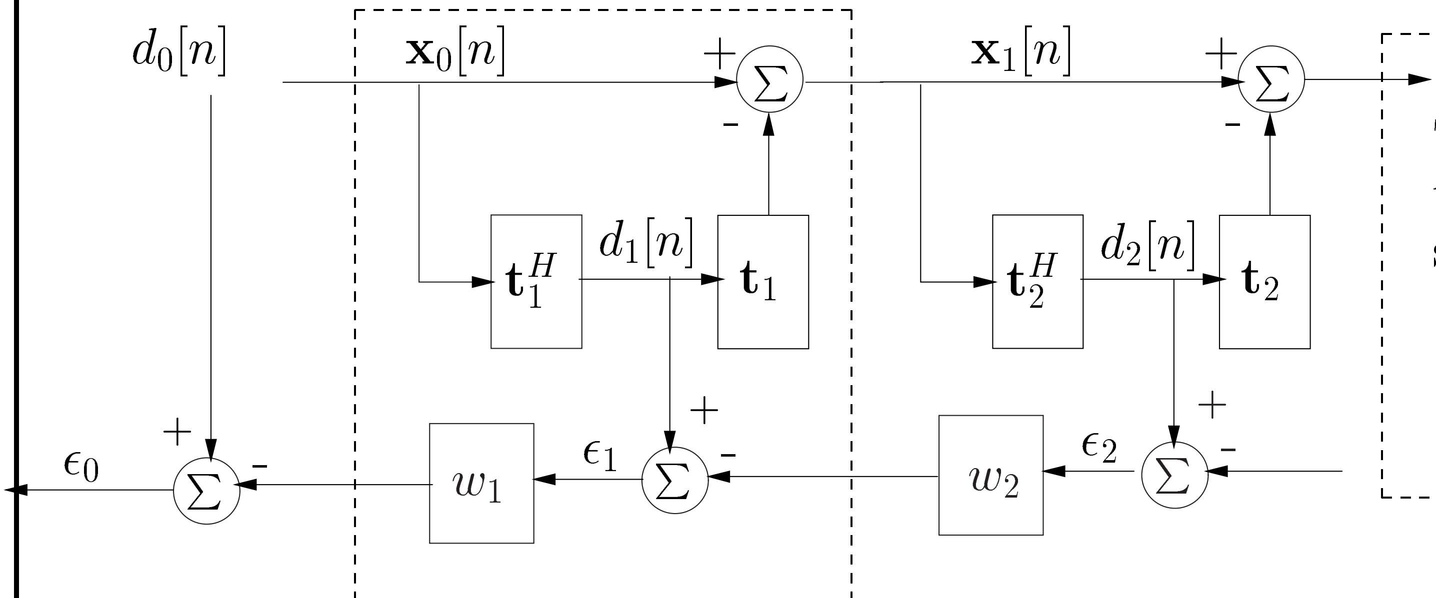
BER for Adaptive Chip-level Equalizers for CDMA Downlink.

Recent fundamental advances on MSNWF

- Ricks and Goldstein showed that the MSNWF can be implemented without blocking matrices
  - further reduces computational complexity of MSNWF relative to full-rank RLS or PC based reduced-rank adaptive filtering
- Ricks and Goldstein developed a lattice/modular MSNWF structure facilitating efficient data-level implementation, alternative to covariance level processing. Advantages of avoiding formation of covariance matrix:
  - reduces computationally complexity
  - facilitates real-time implementation
  - there may not be enough sample support to form a reliable covariance matrix estimate, especially when data vector is high-dimensional and/or the signal statistics are rapidly time-varying
- Honig and Xiao have proven that stopping MSNWF at stage D constrain $\mathbf{w_0}$ to lie in D-dimensional Krylov subspace

$$\mathbf{w_0} \in range\{\mathbf{r_{x_0,d_0}}, \mathbf{R_{x_0}} \mathbf{r_{x_0,d_0}}, \ldots, \mathbf{R}_{\mathbf{x_0}}^{D-1} \mathbf{r_{x_0,d_0}}\}$$

Data-Level Lattice Implementation of MSNWF



The filter can be truncated at any stage $D \leq M - 1$.

Data-Level Lattice Implementation of MSNWF

Forward Recursion:
for $i = 1, ..., D$
$\mathbf{t}_i = \sum_{n=0}^{M-1} d_{i-1}^*[n]\mathbf{x}_{i-1}[n], \quad \mathbf{t}_i = \mathbf{t}_i / \|\mathbf{t}_i\|_2$
$d_i[n] = \mathbf{t}_i^H \mathbf{x}_{i-1}[n], \ \ n = 0, .., M-1$
$\mathbf{x}_i[n] = \mathbf{x}_{i-1}[n] - d_i[n]\mathbf{t}_i, \ \ n = 0, .., M-1$
$\epsilon_D[n] = d_D[n]$
Backwards Recursion:
for $i = (D-1), ..., 1$
$w_{i+1} = \left\{ \sum_{n=0}^{M-1} d_i[n]\epsilon_{i+1}^*[n] \right\} / \left\{ \sum_{n=0}^{M-1} \lvert \epsilon_{i+1}[n] \rvert^2 \right\}$
$\epsilon_i[n] = d_i[n] - w_{i+1}\epsilon_{i+1}[n], \ \ n = 0, .., , M-1$
$\mathbf{w}_0^{(D)} = \sum_{i=1}^{D} (-1)^{i+1} \left\{ \prod_{\ell=1}^{i} w_i \right\} \mathbf{h}_i$

Recent Advances on MSNWF by Zoltowski/Goldstein I

- Developed computationally efficient scheme for generating *orthogonal* basis for Krylov subspace spanned by $\{\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}, \mathbf{R}_{\mathbf{x_0}}\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}, \ldots, \mathbf{R}_{\mathbf{x_0}}^{D-1}\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}\}$:
- each successive member of basis is generated by multiplying previous member by $\mathbf{R}_{\mathbf{x_0}}$ and subtracting off from resulting vector its components onto only last two members of basis:
  - At i-th stage, first compute: $\mathbf{u}_i = \mathbf{R}_{\mathbf{x_0}}\mathbf{t}_{i-1}$
  - next basis vector then computed as:

$$\mathbf{t}_i = \mathbf{u}_i - (\mathbf{t}_{i-1}^H\mathbf{u}_i)\mathbf{t}_{i-1} - (\mathbf{t}_{i-2}^H\mathbf{u}_i)\mathbf{t}_{i-2}$$

  - scale $\mathbf{t}_i$ to have unit norm
- resulting orthogonal basis is identical to that generated via original forward recursion of MSNWF
  - tri-diagonalizes $\mathbf{R}_{\mathbf{x_0}}$ at any stage,
  - computed without need for forming blocking matrices

Recent advances on MSNWF by Zoltowski/Goldstein II

- Developed simple order-recursion for updating weight vector and MSE as each new stage is added
- *recall* original MSNWF composed of two parts: forward recursion followed by backward recursion
- Important to monitor Mean Square Error (MSE) as each new stage is added (new stage $\equiv$ additional basis vector from forward recursion)
  - sample support may be insufficient to support an additional stage such that addition of such may cause MSE to increase
- Since backwards recursion coefficients completely change each time a new basis vector is added, evaluation of its impact on MSE previously required a backwards recursion for each new added stage
- Order-recursive MSNWF allows MSE to be updated at each stage along with backwards-recursion coefficients via a simple recursion
  - facilitates development of statistically based stopping rules

Order-Recursive MSNWF

- At stage D, the orthogonal basis $\mathbf{T}^{(D)} = [\mathbf{t}_1, \ldots, \mathbf{t}_D] \in \mathbf{C}^{N \times D}$ from forward recursion yields length D observation

$$\mathbf{d}^{(D)}[n] = \mathbf{T}^{(D),H}\mathbf{x}_0[n] \in \mathbf{C}^D,$$

- $\mathbf{d}^{(D)}[n]$ has $D \times D$ tri-diagonal covariance matrix:

$$\mathbf{R}_{\mathbf{d}}^{(D)} = \mathcal{E}\{\mathbf{d}^{(D)}[n]\mathbf{d}^{(D),H}[n]\} = \mathbf{T}^{(D),H}\mathbf{R}_{\mathbf{x_0}}\mathbf{T}^{(D)}.$$

- Terminating at stage D, backwards recursion coefficients are components of WF $\mathbf{w}_{\mathbf{d}}^{(D)}$ that estimates $d_0[n]$ from $\mathbf{d}^{(D)}[n]$:

$$\mathbf{w}_{\mathbf{d}}^{(D)} = \left(\mathbf{R}_{\mathbf{d}}^{(D)}\right)^{-1}\mathbf{r}_{\mathbf{d},\mathbf{d_0}}^{(D)} = \left(\mathbf{T}^{(D),H}\mathbf{R}_{\mathbf{x_0}}\mathbf{T}^{(D)}\right)^{-1}\mathbf{T}^{(D),H}\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}$$

- Rank D MSNWF WF:

$$\mathbf{w}_0^{(D)} = \mathbf{T}^{(D)}\mathbf{w}_{\mathbf{d}}^{(D)} = \mathbf{T}^{(D)}\left[\left(\mathbf{T}^{(D),H}\mathbf{R}_{\mathbf{x_0}}\mathbf{T}^{(D)}\right)^{-1}\mathbf{T}^{(D),H}\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}\right]$$

- MSE at stage D:

$$\mathrm{MSE}^{(D)} = \sigma_{d_0}^2 - \mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}^H\mathbf{w}_0^{(D)}$$

Order-Recursive MSNWF

- Rank D MSNWF WF:

$$\mathbf{w}_0^{(D)} = \mathbf{T}^{(D)}\mathbf{w}_{\mathbf{d}}^{(D)} = \mathbf{T}^{(D)}\left(\mathbf{T}^{(D),H}\mathbf{R}_{\mathbf{x_0}}\mathbf{T}^{(D)}\right)^{-1}\mathbf{T}^{(D),H}\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}.$$

- MSE at stage D: $\mathrm{MSE}^{(D)} = \sigma_{d_0}^2 - \mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}^H \mathbf{w}_0^{(D)}$
- *Goal:* update both backwards recursion coefficients $\mathbf{w}_{\mathbf{d}}^{(D)}$ (change with each new stage) and $\mathrm{MSE}^{(D)}$ in terms of $\mathbf{w}_{\mathbf{d}}^{(D-1)}$ and $\mathrm{MSE}^{(D-1)}$
- Recall: $\mathbf{d}^{(D)}[n] = \mathbf{T}^{(D),H}\mathbf{x}_0[n]$ has tri-diagonal covariance matrix:

$$\mathbf{R}_{\mathbf{d}}^{(D)} = \mathbf{T}^{(D),H}\mathbf{R}_{\mathbf{x_0}}\mathbf{T}^{(D)} = \left[\begin{array}{c|c} \mathbf{T}^{(D-1),H}\mathbf{R}_{\mathbf{x_0}}\mathbf{T}^{(D-1)} & \begin{array}{c}\mathbf{0}\\ r_{D-1,D}\end{array} \\ \hline \mathbf{0}^T \quad r_{D-1,D}^* & r_{D,D}\end{array}\right] \in \mathbf{C}^{D\times D}$$

- Given $\mathbf{R}_{\mathbf{d}}^{(D-1)}$ from stage $D-1$, new entries of $\mathbf{R}_{\mathbf{d}}^{(D)}$ are:

$$r_{D-1,D} = \mathbf{t}_{D-1}^H\mathbf{R}_{\mathbf{x_0}}\mathbf{t}_D \quad \text{and} \quad r_{D,D} = \mathbf{t}_D^H\mathbf{R}_{\mathbf{x_0}}\mathbf{t}_D.$$

- only first element of $\mathbf{r}_{\mathbf{d},\mathbf{d_0}}^{(D)}$ is nonzero $\Rightarrow$ only first column of inverse of $\mathbf{R}_{\mathbf{d}}^{(D)}$ is needed $\Rightarrow$ use matrix inversion lemma

$\mathbf{t}_0 = \mathbf{0}, \quad \mathbf{t}_1 = \mathbf{r}_{\mathbf{x_0},\mathbf{d_0}} / \lVert \mathbf{r}_{\mathbf{x_0},\mathbf{d_0}} \rVert_2$
$\mathbf{u} = \mathbf{R}_{\mathbf{x_0}} \mathbf{t}_1$
$r_{0,1} = 0, \quad r_{1,1} = \mathbf{t}_1^H \mathbf{u}$
$c_{\text{first}}^{(1)} = r_{1,1}^{-1}, \quad c_{\text{last}}^{(1)} = r_{1,1}^{-1}$
$\text{MSE}^{(1)} = \sigma_{d_0}^2 - \lVert \mathbf{r}_{\mathbf{x_0},\mathbf{d_0}} \rVert_2^2 c_{\text{first}}^{(1)}$
$\mathbf{v} = \mathbf{u} - r_{i-1,i-1} \mathbf{t}_{i-1} - r_{i-2,i-1} \mathbf{t}_{i-2}$
$r_{i-1,i} = \lVert \mathbf{v} \rVert_2$
$\mathbf{t}_i = \mathbf{v} / r_{i-1,i}$
$\mathbf{u} = \mathbf{R}_{\mathbf{x_0}} \mathbf{t}_i$
$r_{i,i} = \mathbf{t}_i^H \mathbf{u}$
$\beta_i = r_{i,i} - \lvert r_{i-1,i} \rvert^2 c_{\text{last},i-1}^{(i-1)}$
$\mathbf{c}_{\text{first}}^{(i)} = \begin{bmatrix} \mathbf{c}_{\text{first}}^{(i-1)} \\ 0 \end{bmatrix} + \beta_i^{-1} c_{\text{last},1}^{(i-1),*} \begin{bmatrix} \lvert r_{i-1,i} \rvert^2 \mathbf{c}_{\text{last}}^{(i-1)} \\ -r_{i-1,i}^* \end{bmatrix}$
$\mathbf{c}_{\text{last}}^{(i)} = \beta_i^{-1} \begin{bmatrix} -r_{i-1,i} \mathbf{c}_{\text{last}}^{(i-1)} \\ 1 \end{bmatrix}$
$\text{MSE}^{(i)} = \sigma_{d_0}^2 - \lVert \mathbf{r}_{\mathbf{x_0},\mathbf{d_0}} \rVert_2^2 c_{\text{first},1}^{(i)}$
$\mathbf{T}^{(D)} = [\mathbf{t}_1, \ldots, \mathbf{t}_D]$
$\mathbf{w}_0^{(D)} = \mathbf{T}^{(D)} \mathbf{c}_{\text{first}}^{(D)}$

Table 1. Covariance-Level Order-Recursive MSNWF.

Data-Level Order-Recursive MSNWF

- Proposed data-level, order-recursive MSNWF offers following benefits:
  - updates backwards recursion coefficients and MSE at each stage
  - avoids computation of blocking matrices
  - avoids computation of a covariance matrix (for which there may not be sufficient sample support)

for $i = 1, \ldots, D$
$\mathbf{t}_i = \sum_{n=0}^{M-1} d_{i-1}^*[n]\mathbf{x}_{i-1}[n], \quad \mathbf{t}_i = \mathbf{t}_i / \\|\mathbf{t}_i\\|_2$
$d_i[n] = \mathbf{t}_i^H \mathbf{x}_{i-1}[n], \ \ n = 0, .., M-1$
$\mathbf{x}_i[n] = \mathbf{x}_{i-1}[n] - d_i[n]\mathbf{t}_i, \ \ n = 0, .., M-1$
$r_{i-1,i} = \sum_{n=0}^{M-1} d_{i-1}^*[n] d_i[n]$
$r_{i,i} = \sum_{n=0}^{M-1} \lvert d_i[n] \rvert^2$
$\beta_i = r_{i,i} - \lvert r_{i-1,i} \rvert^2 c^{(i-1)}_{\text{last},i-1}$
$\mathbf{c}^{(i)}_{\text{first}} = \begin{bmatrix} \mathbf{c}^{(i-1)}_{\text{first}} \\\\ 0 \end{bmatrix} + \beta_i^{-1} c^{(i-1),*}_{\text{last},1} \begin{bmatrix} \lvert r_{i-1,i} \rvert^2 \mathbf{c}^{(i-1)}_{\text{last}} \\\\ -r^*_{i-1,i} \end{bmatrix}$
$\mathbf{c}^{(i)}_{\text{last}} = \beta_i^{-1} \begin{bmatrix} -r_{i-1,i} \mathbf{c}^{(i-1)}_{\text{last}} \\\\ 1 \end{bmatrix}$
$\text{MSE}^{(i)} = \sigma^2_{d_0} - \\|\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}\\|_2^2 c^{(i)}_{\text{first},1}$
$\mathbf{T}^{(D)} = [\mathbf{t}_1, \ldots, \mathbf{t}_D]$
$\mathbf{w}_0^{(D)} = \mathbf{T}^{(D)} \mathbf{c}^{(D)}_{\text{first}}$

Table 2. Data-Level Order-Recursive MSNWF.

Recent advances on MSNWF by Zoltowski/Goldstein III

- Recently discovered connection between MSNWF and conjugate gradient (CG) search method
- Inherent relationship between MSNWF and CG follows from connection between MSNWF and Krylov subspace estimation
  - At each iteration, CG minimizes $\mathbf{w}^H \mathbf{R}_{\mathbf{x_0}} \mathbf{w} + \mathbf{w}^H \mathbf{r}_{\mathbf{x_0},\mathbf{d_0}} + \mathbf{r}^H_{\mathbf{x_0},\mathbf{d_0}} \mathbf{w}$ over Krylov subspace generated by $\mathbf{R}_{\mathbf{x_0}}$ and $\mathbf{r}_{\mathbf{x_0},\mathbf{d_0}}$
- The fact that an iterative search algorithm is related to a reduced-rank adaptive filtering scheme is surprising!
- Substituting expression for $\mathbf{c}^{(i)}_{\text{first}}$ into $\mathbf{w}_0^{(i)} = \mathbf{T}^{(i)} \mathbf{c}^{(i)}_{\text{first}}$. where $\mathbf{T}^{(i)} = [\mathbf{t}_1, \ldots, \mathbf{t}_i]$, yields a stage to stage direct update of weight vector

$$\mathbf{w}_0^{(i)} = \mathbf{w}_0^{(i-1)} + \gamma_i \mathbf{g}_i + \phi_i \mathbf{t}_i$$

- where:

$$\mathbf{g}_i = \mathbf{T}^{(i)} \mathbf{c}^{(i)}_{\text{last}} = \eta_i \mathbf{g}_{i-1} + \zeta_i \mathbf{t}_i$$

Idea for Further Research: Reduced-Rank DFE

- Reduced-rank Decision Feedback Equalizer (DFE) based on MSWNF
- Finite alphabet constraint on information symbols has heretofore not been exploited in the MSNWF
  - except in decision-directed mode, but even that is not straightforward since MSWNF operates in block-processing mode.
- One proposed scheme for a reduced-rank DFE is a two-stage process:
  - a training sequence is first processed novelly by the MSNWF in such a way as to produce a low-rank estimate of a sparse channel
  - covariance level version of the MSNWF is then used in the second stage to efficiently solve large dimension Wiener-Hopf equations, where the "data" includes past symbol estimates

Selected Idea for Further Research: MUD

- Efficient implementation of multiple MSNWF's running in parallel for Multi-User Detection (MUD) employing parallel interference cancellation
- Each MSNWF provides a reduced-rank MMSE estimate of the respective signal from one of the multiple users
- Enhance interference suppression capability of any one MSNWF by subtracting estimates of the other users' signals based on either hard or soft decisions on their respective symbols $\Rightarrow$ requires communication amongst the multiple parallel MSNWF's
- Incorporate recent developments on iterative/Turbo MUD by Poor et al. to diminish performance gap between single-user MMSE and multi-user MMSE