A NON-UNITARY EXTENSION TO SPECTRAL ESTIMATION

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ABSTRACT

This paper introduces a new parametric frequency estimation technique, ROCK MUSIC, for uncovering frequency components of a sum of complex exponentials in noise. The ROCK MUSIC method is similar to conventional frequency estimation methods, like MUSIC, that decompose the autocorrelation matrix into signal and noise subspaces. The distinguishing feature of this new method is that it does not use an eigenvector decomposition but a decomposition based on a **R**educed **O**rder **C**orrelation **K**ernel.

This new non-unitary basis compresses the signal space, thereby making it much more robust to incorrect subspace divisions than previous subspace frequency estimation methods. Simulation of the ROCK MUSIC technique indicates that large signal spaces may be represented by just a few basis vectors. The significance of this finding is that one does not need to know the correct number of signals present in order to obtain classification of the frequencies. Thus the new ROCK MUSIC method resolves a fundamental limitation of conventional subspace frequency estimation techniques - the requirement that the number of signals or complex exponentials be known before their frequencies can be estimated.

1. INTRODUCTION

An important problem in spectral estimation theory is to identify which frequencies are present in a process which is known to be a sum of complex exponentials in white noise. This problem is frequently encountered in the spatial domain where it is desired to determine the angle of arrival of the plane waves impinging a linear antenna array. In this context, the arrival angles of the plane waves onto the array represent the frequencies to be estimated. A common spectrum estimation technique is to exploit the known parametric form of the process to estimate which frequencies are present by dividing the space spanned by the input data into signal and noise subspaces.

2. BACKGROUND

The method referred to as Pisarenko's harmonic decomposition [1] divides the data space into orthogonal signal and noise subspaces and then projects a steering vector $\mathrm{s}(\theta)$ onto the noise subspace in order to estimate those frequencies present. Pisarenko's method requires that the exact number of complex exponentials, p, be known in order to correctly partition the data space. Let the vector random process $\mathbf{x}$ represent the observed data, which is assumed to be zero-mean without loss in generality. The $(p+1)\times(p+1)$ autocorrelation matrix $\mathbf{R}_x = \mathbf{E}[\mathbf{x}\mathbf{x}^H]$ is then computed and decomposed into its eigenvectors and eigenvalues. The p eigenvectors associated with the p largest eigenvalues span the signal space and the eigenvector associated with the smallest eigenvalue, $\mathbf{v}_{min}$, represents the noise space.

The Pisarenko frequency estimation function,

$$\mathbf{S}_{pisarenko}(\theta) = \frac{1}{\parallel \mathbf{v}_{min}^H \mathrm{s}(\theta) \parallel^2}, \tag{1}$$

utilizes the orthogonality of the noise and signal space. The inner product of the noise space vector $\mathbf{v}_{min}$ with the steering vector $\mathrm{s}(\theta)$ is zero for all those look directions which correspond with the signal space. Therefore the frequency estimation function will have a peak at each of the signal frequencies. Pisarenko's method has a number of drawbacks which make it difficult to use in practice. A practical limitation is that the exact number of complex exponentials must be known prior to estimating their frequencies. In many applications the number of signals present is unknown and this would prevent the use of Pisarenko's method. Another problem with implementing Pisarenko's method is the computational requirements for finding the eigenvectors of the data matrix in high-order problems.

An improvement to Pisarenko's method of parametric frequency estimation is called MUSIC for MUltiple SIgnal Classification [1, 2]. Music differs from Pisarenko's method in that the auto-correlation matrix $\mathbf{R}_x$ is no longer limited to rank p+1. Instead MUSIC requires that the correlation matrix $\mathbf{R}_x$ now be of order $N > p$, where p is the number of signals present. This larger autocorrelation matrix is decomposed into its eigenvectors and eigenvalues, and the eigenvectors associated with the largest p eigenvalues are assumed to span the signal space. This implies that the noise space is now of dimension N-p. Therefore, for each noise eigenvector there will be p zeros which lie on the unit circle and an additional N-p-1 zeros which can lie anywhere including close to the unit circle. These additional zeros can give rise to spurious peaks which make it difficult to distinguish between the noise related peaks and the true signal peaks. This is not a problem with Pisarenko's method because there is only one noise vector.

In order to mitigate the spurious peaks caused by the additional dimensions of the noise, the MUSIC algorithm averages the noise eigenvectors using the frequency estimation function,

$$\mathbf{S}_{music}(\theta) = \frac{1}{\sum_{i=N-p}^{N} \| \mathbf{v}_i^H \mathbf{s}(\theta) \|^2}, \quad (2)$$

where $\mathbf{s}(\theta)$ is projected onto $\{\mathbf{v}_i\}$, the set of eigenvectors associated with the N-p smallest eigenvalues of $\mathbf{R}_x$, and summed together. Although MUSIC is an improvement over Pisarenko's method because it allows some flexibility in overestimating the signal space rank, it has very poor performance if the signal space is underestimated. MUSIC therefore still requires knowledge of how many signals are present in order to make an estimate of their frequencies. Finally, MUSIC also requires the computational burden of computing the eigenvectors $\mathbf{R}_x$.

3. ROCK MUSIC

The new ROCK MUSIC frequency estimation technique uses a decomposition based on a **R**educed **O**rder **C**orrelation **K**ernel [3] to replace the eigenvector decomposition in the classical MUSIC algorithm. The ROCK MUSIC decomposition defines a new, generally non-unitary, diagonalizing basis to span the data space. This new basis forms a spectral decomposition that compresses the signal space and therefore requires very few basis vectors to represent a complex signal environment. By selecting a small number, b, of basis vectors, the new ROCK MUSIC frequency estimation function is

$$\mathbf{S}_{rock\ music}(\theta) = \frac{1}{\sum_{i=N-b}^{N} \| \mathbf{q}_i^H \mathbf{s}(\theta) \|^2}, \quad (3)$$

where $\mathbf{q}_i$ is the i-th column of matrix $\mathbf{Q}_N$ composed of the new non-unitary basis vectors to be defined. The ROCK MUSIC technique can effectively estimate the signal space in a manner which is nearly independent of the number of complex exponentials present. Thus ROCK MUSIC resolves a fundamental limitation of previous parametric frequency estimation techniques.

ROCK MUSIC's non-unitary basis set is created in a stage by stage manner. It is necessary to assume exactly one frequency is known *a priori* in order to initialize the ROCK MUSIC algorithm. This pivot into the algorithm can correspond to a frequency which is known to be present, known to be absent, or whose status is unknown and will require later testing (see Sect. 5). This known frequency corresponds to a normalized signal vector which is denoted $\mathbf{h}_1$. The matrix $\mathbf{B}_1$ is then defined to be the operator which projects onto the nullspace of $\mathbf{h}_1$. The ROCK MUSIC algorithm begins by projecting the observed data $\mathbf{x}$ using the operators $\mathbf{h}_1$ and $\mathbf{B}_1$:

$$d_1 = \mathbf{h}_1^H \mathbf{x}, \quad (4)$$

$$\mathbf{x}_1 = \mathbf{B}_1 \mathbf{x}. \quad (5)$$

To complete the initialization, the first-stage cross-correlation vector is defined by,

$$\mathbf{r}_{x_1 d_1} = \mathbf{B}_1 \mathbf{R}_x \mathbf{h}_1. \quad (6)$$

At each subsequent stage ROCK MUSIC projects the data onto two subspaces. One subspace points in the direction of a normalized cross-correlation vector h_i and the other subspace spans the space orthogonal to that cross-correlation vector, created by projecting the data with the operator B_i, where $B_i = null(h_i)$ (see Fig. 1). The recursions used are as follows:

$$\mathbf{r}_{x_i d_i} = \mathbf{B}_i \mathbf{R}_{x_{i-1}} \mathbf{h}_i, \qquad i \geq 2, \quad (7)$$

$$\mathbf{h}_{i+1} = \frac{\mathbf{r}_{x_i d_i}}{\| \mathbf{r}_{x_i d_i} \|}, \qquad i \geq 2. \quad (8)$$

In this fashion the data is decomposed into smaller and smaller subspaces,

$$\mathbf{R}_{x_i} = \left(\prod_{k=i}^{1} \mathbf{B}_k \right) \mathbf{R}_x \left(\prod_{k=1}^{i} \mathbf{B}_k^H \right). \quad (9)$$

The first subspace projections are represented by the operator $\mathbf{L}$ of rank N defined below.

$$\mathbf{L}_N = \begin{bmatrix} \mathbf{h}_1^H \\ \mathbf{h}_2^H \mathbf{B}_1 \\ \vdots \\ \mathbf{h}_{N-1}^H \prod_{i=N-2}^{1} \mathbf{B}_i \\ \prod_{i=N-1}^{1} \mathbf{B}_i \end{bmatrix}. \quad (10)$$

The matrix $\mathbf{L}_N$ is a unitary transformation and tri-diagonalizes the auto-correlation matrix $\mathbf{R}_x$. In order to complete the diagonalization of $\mathbf{R}_x$ a Gram-Schmidt style operator $\mathbf{U}_N$ is applied to the output of $\mathbf{L}_N$ where $\mathbf{U}_N$ is given by,

$$U_N = \begin{bmatrix} 1 & -w_1^* & \cdots & (-1)^{N+1} \prod_{i=1}^{N-1} w_i^* \\ 0 & 1 & \cdots & (-1)^N \prod_{i=2}^{N-1} w_i^* \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -w_{N-1}^* \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \quad (11)$$

The scalar filter weights w_i are found using a backwards recursion of the scalar Wiener-Hopf equation

$$w_i = \xi_{i+1}^{-1} r_{\varepsilon_{i+1} d_i}, \quad (12)$$

where

$$\xi_i = \sigma_{di}^2 - \frac{|\delta_{i+1}|^2}{\xi_{i+1}}, \quad (13)$$

defines the minimum mean square error (MMSE) at each stage, $\sigma_{d_i}^2$ is the variance of the "*desired*" signal d_i, $r_{\varepsilon_{i+1} d_i}$ is the scalar cross-correlation computed between the error from stage i and the desired signal at stage i+1, and $\delta_{i+1} = ||r_{\varepsilon_{i+1} d_i}||$. For the last stage the MMSE is given by $\xi_N = \sigma_{d_N}^2 = \sigma_{x_{N-1}}^2$.

The matrix operator $\mathbf{L}_N$ is intentionally constructed to be unitary while the matrix operator $\mathbf{U}_N$ is generally non-unitary. Thus, the matrix operator $\mathbf{Q}_N = (\mathbf{U}_N \mathbf{L}_N)^H$ is generally non-unitary and diagonalizes the data matrix $\mathbf{R}_x$:

$$\mathbf{\Lambda}_\xi = \begin{bmatrix} \xi_1 & 0 & 0 & 0 \\ 0 & \xi_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \xi_N \end{bmatrix} = \mathbf{Q}_N^H \mathbf{R}_x \mathbf{Q}_N. \tag{14}$$

The matrix $\mathbf{Q}_N$ creates a new non-unitary basis whose column vectors span the data space and can be divided into signal and noise subspaces. For a more complete description of the decomposition of R_x see [4-6] on the multistage Wiener filter. Finally, the reduced-order matrix $\mathbf{Q}_M = (\mathbf{U}_M \mathbf{L}_M)^H$ is obtained by truncating $\mathbf{L}_N$ in (10) to keep the first M out of N rows (forming the $M \times N$ matrix $\mathbf{L}_M$) and then computing the $M \times M$ matrix $\mathbf{U}_M$ in exactly the manner described in (11).

4. SIMULATION RESULTS

The performance of ROCK MUSIC and the classical MUSIC technique are compared for the problem of estimating the angles of arrival for multiple plane waves impinging a uniform line array whose elements are equispaced at half-wavelength. The robustness of the algorithm's angle classification performance are evaluated with respect to the signal rank. To quantify the results, let any peak above 30dB at a true angle of arrival be counted as a correct identification. Due to the finite resolution of the sensor array some angles are identified in the same peak. The angles are selected from a uniform random distribution between -90 and 90 degrees. The simulations are all run for a sensor array of 128 elements with 40 plane waves impinging on the array. A total of 150 data snapshots are used to estimate the auto-covariance matrix $\mathbf{R}_x$.

Figs. 2 and 3 show the baseline performance when both algorithms are given 40 basis vectors to span the signal space created by 40 plane waves impinging on the array. These figures illustrates that, under these conditions, both methods perform very well, correctly identifying all 40 arrival angles of the plane waves.

Figs. 4 and 5 show the performance when both algorithms are given 30 basis vectors to define the signal space composed of 40 plane waves impinging on the array. In this case, MUSIC fails to identify a single angle of arrival correctly, whereas ROCK MUSIC's perfect performance is unaffected. This example illustrates the limitations of the MUSIC algorithm to classify the angles of arrival when the number of signals impinging on the array is underestimated. It demonstrates that MUSIC needs the full rank of the signal space in order to adequately represent it.

Figs. 6 and 7 show the performance of the algorithms when only 3 basis vectors are used to define the signal space. Here again MUSIC fails to classify a single angle of arrival correctly where as ROCK MUSIC's performance is still 100% accurate.

5. CONCLUSION

One interesting requirement of the ROCK MUSIC algorithm is the selection of a pivot or initial signal vector $\mathbf{h}_1$. A signal is always found at the frequency determined by $\mathbf{h}_1$, even if there is no signal actually present. However ROCK MUSIC consistently identifies the remaining arrival angles correctly for the other plane waves impinging on the array. This shows an ambiguity of the ROCK MUSIC algorithm to identify the presence of a signal at that initial look angle. Solutions to this situation include determining one signal vector beforehand (perhaps the most powerful), picking one frequency in which it is known that no signal vector can be present and then ignoring the false peak, or testing for the presence of the pivot signal afterwards. Further research is underway to resolve this ambiguity.

This paper introduces a new parametric frequency estimation technique based on a novel non-unitary basis which compresses the signal space. Simulation shows ROCK MUSIC is able to identify the angles of arrival without knowledge of the exact rank of the signal space. Thus ROCK MUSIC overcomes a fundamental limitation of conventional parametric frequency estimation methods which require that the signal subspace size be known exactly in order to operate. The compressed signal space created by the novel non-unitary basis also promises to need less data samples to estimate the auto-correlation matrix $\mathbf{R}_x$. Finally, the reduced signal space allows computational savings gained from only having to calculate a few basis vectors to define the signal space.

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