1. Let a real-valued D-T signal \( x(n) \) be decomposed as \( x(n) = x_e(n) + x_o(n) \), where \( x_e(n) \) and \( x_o(n) \) are the even and odd parts of \( x(n) \), respectively, defined as

\[
x_e(n) = \frac{1}{2} \{ x(n) + x(-n) \}
\]

\[
x_o(n) = \frac{1}{2} \{ x(n) - x(-n) \}
\]

Let \( X_e(e^{j\omega}) \) and \( X_o(e^{j\omega}) \) be the DTFT’s of \( x_e(n) \) and \( x_o(n) \), respectively.

(a) Prove the following two relationships

\[
X_e(e^{j\omega}) = \text{Re}\{X(e^{j\omega})\}
\]

\[
X_o(e^{j\omega}) = j \text{Im}\{X(e^{j\omega})\}
\]

(b) Let \( u_e(n) \) and \( u_o(n) \) be the even and odd parts of the unit step sequence \( u(n) \). Determine the DTFT of \( u_e(n) \), denoted \( U_e(e^{j\omega}) \), and determine the DTFT of \( u_o(n) \), denoted \( U_o(e^{j\omega}) \).

(c) Combine \( U_e(e^{j\omega}) \) and \( U_o(e^{j\omega}) \) based on the results proved in (a) to obtain an expression for the DTFT of \( u(n) \), denoted \( U(e^{j\omega}) \). Show that

\[
U(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi \delta(\omega - 2\pi k)
\]

2. Consider the causal LTI system described by the difference equation

\[
y(n) = 2\cos \omega_o y(n-1) - y(n-2) + A \sin \omega_o x(n)
\]

where \( A \) and \( \omega_o \) are just fixed constants with \( 0 < \omega_o < \pi \).

(a) Determine an expression for the frequency response \( H(e^{j\omega}) \) of the system and provide a rough sketch of \( |H(e^{j\omega})| \). Specify the value of \( |H(e^{j\omega})| \) \( \omega = \omega_o \).

(b) Show that the impulse response of the system is \( h(n) = A \sin[\omega_o(n+1)]u(n) \). You may do this by simply letting \( x(n) = \delta(n) \) in the difference equation above and showing that \( y(n) = A \sin[\omega_o(n+1)] \) for \( n = 0, 1, 2 \). This system is a digital sinusoidal oscillator.
3. Let $x(n)$ and $y(n)$ be two D-T signals and $X(e^{j\omega})$ and $Y(e^{j\omega})$ their respective DTFT’s.

(a) Defining $v(n)$ as the product of $x(n)$ and $y(n)$, i.e., $v(n) = x(n)y(n)$, show that the DTFT of $v(n)$ may be determined from $X(e^{j\omega})$ and $Y(e^{j\omega})$ according to

$$X(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\mu})Y(e^{j(\omega-\mu)}) d\mu$$

(b) Using the result above determine the DTFT, $X(e^{j\omega})$, of the D-T signal

$$x(n) = \frac{\sin^2(\pi n/4)}{\pi^2 n^2}$$

4. Let $x(n)$ and $y(n)$ be two D-T signals and $X(e^{j\omega})$ and $Y(e^{j\omega})$ their respective DTFT’s

(a) By using the convolution theorem and other appropriate DTFT properties, determine and expression for the D-T signal whose DTFT is $X(e^{j\omega})Y^*(e^{j\omega})$.

(b) By using the result in part (a), show that

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega}) d\omega$$

(c) By using the result in part (b), determine the numerical value of the sum

$$\sum_{n=-\infty}^{\infty} \frac{\sin(\pi n/4)}{2\pi n} \frac{\sin(\pi n/6)}{5\pi n}$$

5.

(a) Let the D-T signal $x(n)$ be obtained by sampling the C-T signal

$$x(t) = \text{sinc}^2(100t)$$

at a rate $f_s$ samples/sec. Let $X(e^{j\omega})$ denote the DTFT of $x(n)$.

(i) Plot $|X(e^{j\omega})|$ when $f_s = 800$ Hz.
(ii) Plot $|X(e^{j\omega})|$ when $f_s = 400$ Hz.
(iii) Plot $|X(e^{j\omega})|$ when $f_s = 100$ Hz.

(b) Repeat the above for

$$x(t) = \text{sinc}^2(100t)\cos(2\pi 200t)$$