Solutions to Homework #5

EE438 Spring 2001

$\boxed{\#1}$ $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$

(a) $$\frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2 = \frac{1}{N}\sum_{k=0}^{N-1}\left|\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}} \sum_{\ell=0}^{N-1} x^*(\ell) e^{j\frac{2\pi \ell k}{N}}\right|$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left|\sum_{n=0}^{N-1}\sum_{\ell=0}^{N-1} x(n)x^*(\ell) e^{-j\frac{2\pi k(n-\ell)}{N}}\right|$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{\ell=0}^{N-1} x(n)x^*(\ell)\sum_{k=0}^{N-1} e^{-j\frac{2\pi k(n-\ell)}{N}}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{\ell=0}^{N-1} x(n)x^*(\ell)\,\delta(n-\ell)\,N$$

$$= \sum_{n=0}^{N-1}|x(n)|^2$$

(b) $$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$ ← Regular forward DFT

Replace k with n and n with m to write in terms of $X(n)$.

$$X(n) = \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi nm}{N}}$$

$$\text{DFT}(X(n)) = \sum_{n=0}^{N-1} X(n) e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi n(m+k)}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\sum_{n=0}^{N-1} e^{-j\frac{2\pi n(m+k)}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\sum_{\ell=-\infty}^{\infty} N\,\delta((m+k)-\ell N) = \begin{cases} N x(0), & k=0 \\ N x(N-k), & k=0,\dots,N-1 \end{cases}$$

$= N\,x(-k)$ by periodicity (see problem statement)

2.

a) $$X(k) = \sum_{n=0}^{N-1} x[n]\, W_N^{nk} \quad \text{for } k=0,\dots,N-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n]\, W_N^{nk} + \sum_{n=\frac{N}{2}}^{N-1} x[n]\, W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n]\, W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x[n+\tfrac{N}{2}]\, W_N^{(n+\frac{N}{2})k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] + W_N^{\frac{N}{2}k}\, x[n+\tfrac{N}{2}]\right) W_N^{nk}, \quad k=0,\dots,N-1$$

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] + W_N^{\frac{N}{2}\cdot 2k}\, x[n+\tfrac{N}{2}]\right) W_N^{n2k} \quad \text{for } k=0,\dots,\tfrac{N}{2}-1$$

Note: $W_N^{\frac{N2k}{2}} = W_N^{Nk} = e^{-j\frac{2\pi}{N}\cdot Nk} = e^{-j2\pi k} = 1$

and $W_N^{n2k} = e^{-j\frac{2\pi n 2k}{N}} = e^{-j\frac{2\pi nk}{N/2}} = W_{N/2}^{nk}$

so $$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] + x[n+\tfrac{N}{2}]\right) W_{N/2}^{nk}$$

$$= \underline{\sum_{n=0}^{\frac{N}{2}-1} f_1(n)\, W_{N/2}^{nk} \quad \text{for } k=0,\dots,\tfrac{N}{2}-1}$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] + W_N^{\frac{N}{2}(2k+1)}\, x[n+\tfrac{N}{2}]\right) W_N^{n(2k+1)}, \quad k=0,\dots,\tfrac{N}{2}-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] + \underbrace{W_N^{\frac{N}{2}}}\, x[n+\tfrac{N}{2}]\right) W_N^{n}\, W_N^{2nk}$$

$W_N^{\frac{N}{2}} = e^{-j\pi} = -1$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] - x[n+\tfrac{N}{2}]\right) W_N^{n}\, W_{N/2}^{nk}$$

$$= \underline{\sum_{n=0}^{\frac{N}{2}-1} f_2(n)\, W_{N/2}^{nk} \quad \text{for } k=0,\dots,\tfrac{N}{2}-1}$$

b) For N=8:

$$X(k) = \sum_{n=0}^{7} x[n] e^{-j\frac{2\pi kn}{8}} \quad k=0,\ldots,7$$

Using the results of part (a)

$$X(2k) = \sum_{n=0}^{3} f_1(n) W_4^{nk} \quad k=0,\ldots,3$$

$$X(2k+1) = \sum_{n=0}^{3} f_2(n) W_4^{nk} \quad k=0,1,2,3$$

where $f_1(n) = x(n) + x(n+4)$
$f_2(n) = (x(n) - x(n+4)) W_8^n$

Now repeat for $X(2k)$ and $X(2k+1)$

$$X(2(2k)) = X(4k) = \sum_{n=0}^{1} f_3(n) W_2^{nk} \quad k=0,1 \qquad (1)$$

$$X(2(2k+1)) = X(4k+2) = \sum_{n=0}^{1} f_4(n) W_2^{nk} \quad k=0,1 \qquad (2)$$

where $f_3(n) = f_1(n) + f_1(n+2)$
$f_4(n) = (f_1(n) - f_1(n+2)) W_4^n$

And

$$X(2(2k)+1) = X(4k+1) = \sum_{n=0}^{1} f_5(n) W_2^{nk} \quad k=0,1 \qquad (3)$$

$$X(2(2k+1)+1) = X(4k+3) = \sum_{n=0}^{1} f_6(n) W_2^{nk} \quad k=0,1 \qquad (4)$$

where $f_5 = f_2(n) + f_2(n+2)$
$f_6 = (f_2(n) + f_2(n+2)) W_4^n$

Finally, from (1) we get

$$X(4k)\Big|_{k=0} = X(0) = f_3(0)W_2^{0\cdot 0} + f_3(1)W_2^{1\cdot 0} = f_3(0) + f_3(1)$$

$$X(4k)\Big|_{k=1} = X(4) = f_3(0)W_2^{0\cdot 1} + f_3(1)W_2^{1\cdot 1} = f_3(0) - f_3(1)$$

From (2):

$$X(4k+2)\Big|_{k=0} = X(2) = f_4(0) + f_4(1)$$

$$X(4k+2)\Big|_{k=1} = X(6) = f_4(0) - f_4(1)$$

From (3):

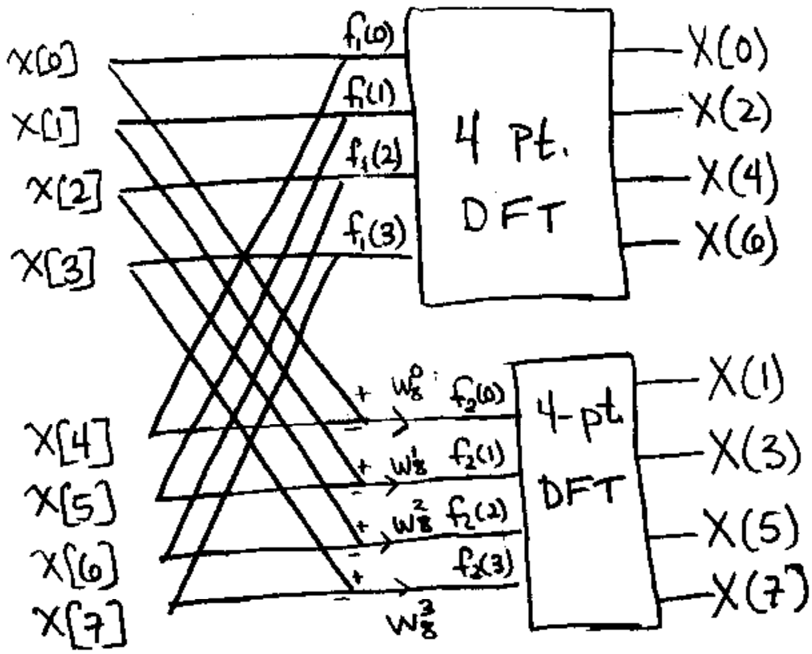
$$X(4k+1)\Big|_{k=0} = X(1) = f_5(0) + f_5(1)$$

$$X(4k+1)\Big|_{k=1} = X(5) = f_5(0) - f_5(1)$$

From (4):

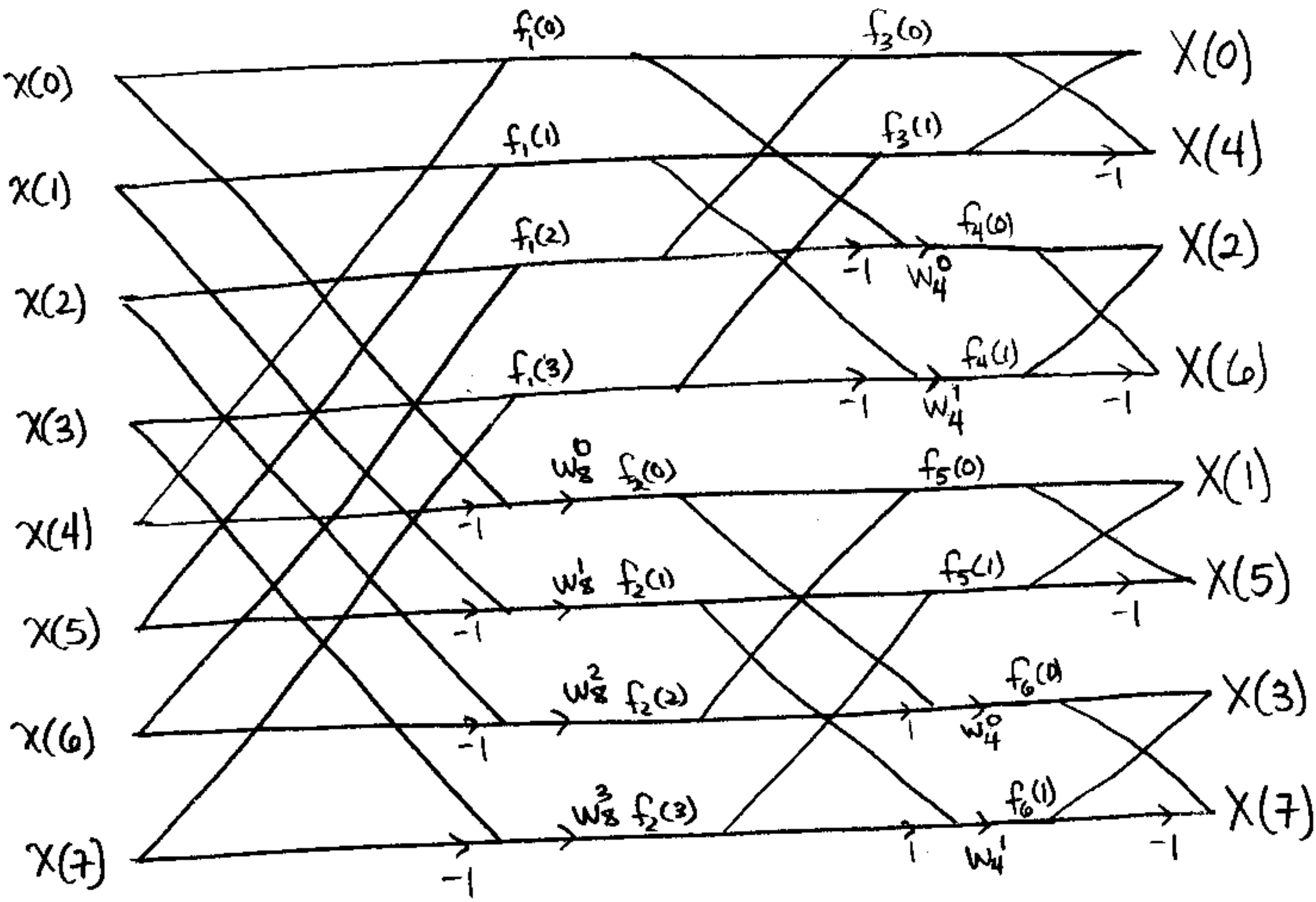
$$X(4k+3)\Big|_{k=0} = X(3) = f_6(0) + f_6(1)$$

$$X(4k+3)\Big|_{k=1} = X(7) = f_6(0) - f_6(1)$$



← First stage

Full diagram ↙



3

a) $P = NT_s$ ← sampling Interval

Pitch period

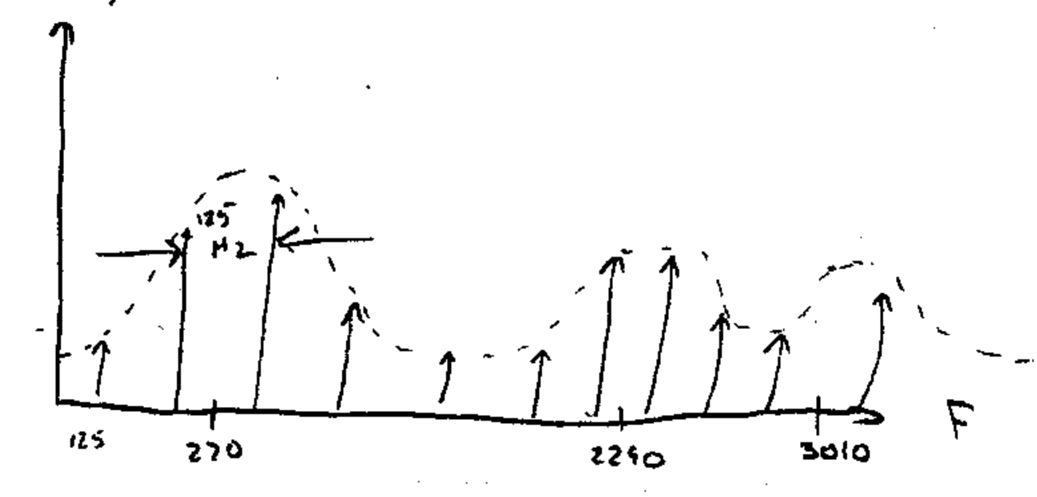
$$N = \frac{1/T_s}{1/P} = \frac{10^4\ Hz}{125\ Hz} = 80 \text{ samples}$$

b) Denominator may be factored as:

$$(z - |P_i| e^{j\theta_i})(z - |P_i| e^{-j\theta_i}) = z^2 - 2|P_i|\cos\theta_i + |P_i|$$

· The angle of a pole may be associated with a formant frequency.

$$\theta_1 = 2\pi \frac{270}{10^4}$$
$$\theta_2 = 2\pi \left(\frac{2290}{10^4}\right)$$
$$\theta_3 = 2\pi \left(\frac{3010}{10^4}\right)$$

c) 

4 $$X_n(e^{j\omega}) = \sum_m x(m)\, w(n-m)\, e^{-j\omega m}$$

(a) $v(n) = a\,x(n) + b\,y(n)$

$$V_n(e^{j\omega}) = \sum_m \left[a\,x(m) + b\,y(m)\right] w(n-m)\, e^{-j\omega m}$$

$$= \sum_m a\,x(m)\, w(n-m)\, e^{-j\omega m} + \sum_m b\,y(m)\, w(n-m)\, e^{-j\omega m}$$

$$= a\,X_n(e^{j\omega}) + b\,Y_n(e^{j\omega})$$

(b) $v(n) = x(n-n_0)$

$$V_n(e^{j\omega}) = \sum_m x(m-n_0)\, w(n-m)\, e^{-j\omega m}$$

let $k = m - n_0$

$$V_n(e^{j\omega}) = \sum_k x(k)\, w(n-(k+n_0))\, e^{-j\omega(k+n_0)}$$

$$= \sum_k x(k)\, w(n-n_0-k)\, e^{-j\omega k}\, e^{-j\omega n_0}$$

$$= X_{n-n_0}(e^{j\omega})\, e^{-j\omega n_0}$$

(c)

$$X_n(e^{j\omega}) = \sum_m x(m)\, w(n-m)\, e^{-j\omega m}$$

$$w(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j\theta})\, e^{j\theta n}\, d\theta$$

$$w(n-m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j\theta})\, e^{j\theta(n-m)}\, d\theta$$

$$\therefore\ X_n(e^{j\omega}) = \sum_m x(m)\, \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j\theta})\, e^{j\theta(n-m)}\, d\theta\, e^{-j\omega m}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j\theta})\, e^{j\theta n} \sum_m x(m)\, e^{-jm(\omega+\theta)}\, d\theta$$

$$\sum_m x(m)\, e^{-jm(\omega+\theta)} = X\left(e^{j(\theta+\omega)}\right)$$

$$\therefore\ = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j\theta})\, e^{j\theta n}\, X\left(e^{j(\theta+\omega)}\right) d\theta$$

(d) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_n(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} x(m)\, w(n-m)\, e^{-j\omega m} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} x(m)\, w(n-m) \int_{-\pi}^{\pi} e^{-j\omega(m-n)}\, d\omega$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} x(m)\, w(n-m) \cdot \delta(n-m)\, 2\pi$$

$$= x(n)\, w(0)$$

$$\therefore\quad x(n)\, w(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_n(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x(n) = \frac{1}{2\pi\, w(0)} \int_{-\pi}^{\pi} X_n(e^{j\omega})\, e^{j\omega n}\, d\omega$$

(5) Formant frequencies: 500 Hz

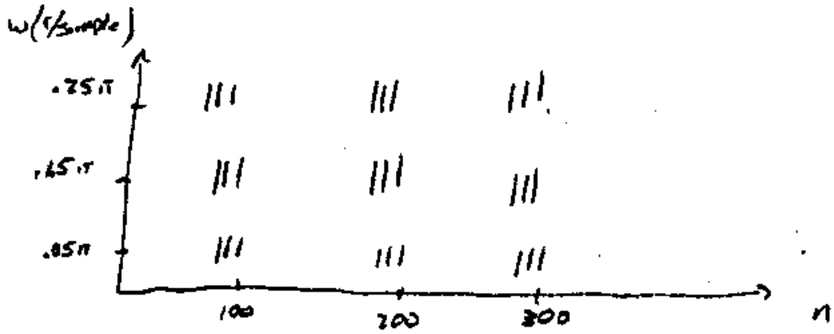
1500 Hz

2500 Hz

pitch period : .005 sec

sampling rate : 20 KHz

(a) N = 50 pts < pitch period => wideband spectrum



Pitch period is .005 x 20,000 = 100 samples

$$f_1 = \frac{2\pi(500)}{2\times10^4} = .05\pi$$

$$f_2 = \frac{2\pi(1.5k)}{2\times10^4} \quad .15\pi$$

$$f_3 = \frac{2\pi(2.5k)}{2\times10^4} \quad .25\pi$$

(b) N = 1000 pts >> pitch period => narrow band

Digital pitch freq $= \frac{2\pi}{100} = 0.02\pi$ r/sample