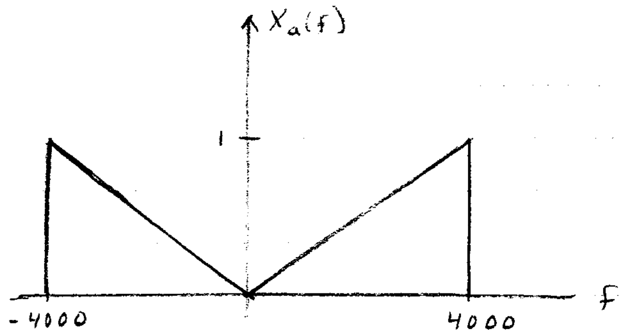
SPRING 2001

EE 438

Assignment # 3

Solutions

1)



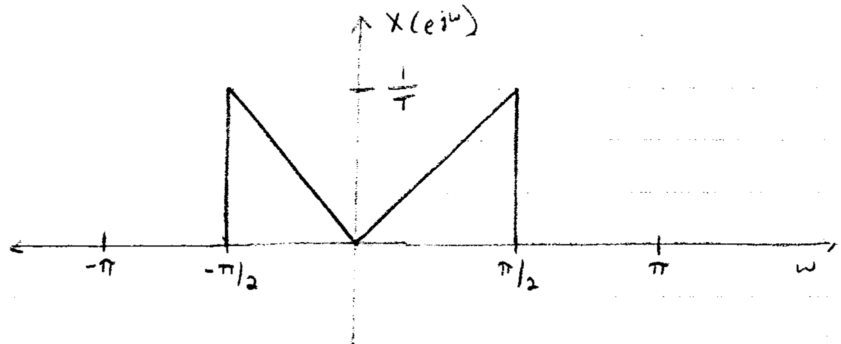
$f_s = 16$ KHz

(a) Sampling frequency goes to 2π in ω domain.

$$f_s = 16 \text{ KHz} \rightarrow 2\pi$$

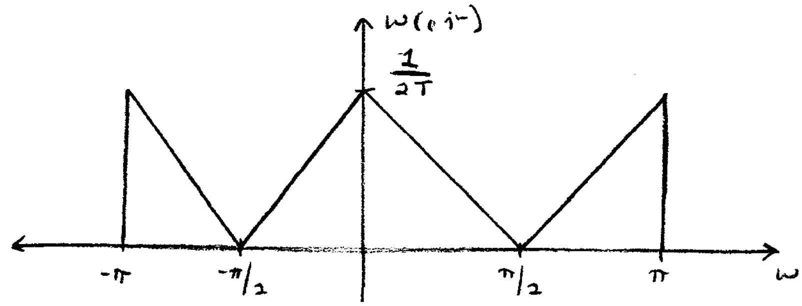
$$\therefore \text{ CUTOFF}(X(e^{j\omega})) = \frac{4 \text{ kHz}}{16 \text{ kHz}}(2\pi) = \pi/2$$

AMPLITUDE IS SCALED BY $f_s = 1/T$

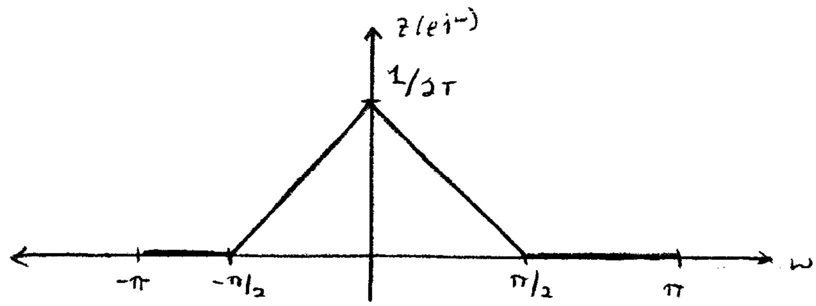


(b) $w(n) = \cos(\pi/2\, n)\, x(n)$

$$W(e^{j\omega}) = \pi \left\{ \delta(\omega - \pi/2) + \delta(\omega + \pi/2) \right\} * X(e^{j\omega})$$
$$= 1/2 \left\{ X(e^{j(\omega - \pi/2)}) + X(e^{j(\omega + \pi/2)}) \right\}$$

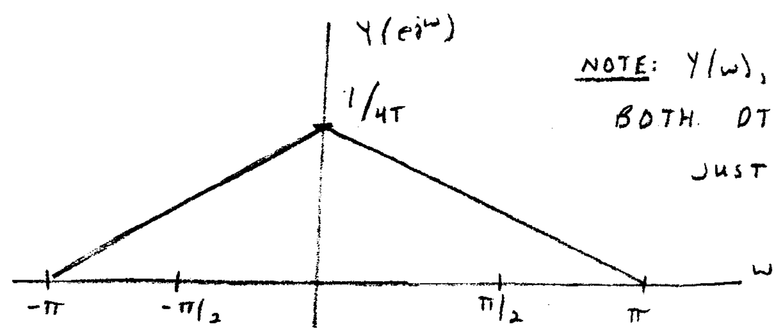


$$Z(e^{j\omega}) = H_{LP}(e^{j\omega})\, W(e^{j\omega}) = \begin{cases} W(e^{j\omega}) & -\pi/2 \le \omega \le \pi/2 \\ 0, & \text{else.} \end{cases}$$



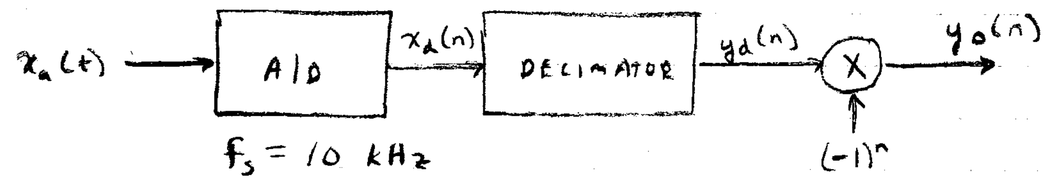
DECIMATOR: $Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} Z\left(\frac{\omega + 2\pi k}{D}\right) \qquad k = 0, 1, \ldots (D-1)$

with $D = 2$, $Y(e^{j\omega}) = \frac{1}{2} Z(e^{j\omega/2}) + \frac{1}{2} Z(e^{j\frac{(\omega - 2\pi)}{2}})$



<u>NOTE</u>: $Y(\omega)$, $Y(e^{j\omega})$ ARE BOTH DTFT$\{y(n)\}$, JUST <u>NOTATION</u>.

2) $x_a(t) = \cos[2\pi(2000t)] + \cos[2\pi(6000)t]$



(a) $f_s \rightarrow 2\pi$

$\therefore \quad f_{a_1} = 2\text{ kHz} \rightarrow \frac{2\text{ kHz}}{10\text{ kHz}}(2\pi) = 2/5\,\pi$ rad/sample

$f_{a_2} = 4\text{ kHz} \rightarrow \frac{4\text{ kHz}}{10\text{ kHz}}(2\pi) = 4/5\,\pi$ rad/sample

Also $-2\pi/5$, $-4\pi/5$

IN RANGE $0 \leq \omega \leq \pi$: $2\pi/5$, $4\pi/5$

(b) $y_d(n) = x_d(2n)$

i.e. just taking every other sample, analogous to $x_a(t)$ sampled at 5 kHz.

$\therefore \quad f_{a_1} = 2\text{ kHz} \rightarrow \frac{2\text{ kHz}}{10\text{ kHz}}(2\pi) = 4\pi/5$ rad/sample

$f_{a_2} = 4\text{ kHz} \rightarrow \frac{4\text{ kHz}}{10\text{ kHz}}(2\pi) - 2\pi = -2\pi/5$ rad/sample

Also $-4\pi/5$, $2\pi/5$

IN RANGE $0 \leq \omega \leq \pi$: $2\pi/5$, $4\pi/5$

NOTE ALIASING DUE TO THE NYQUIST CRITERIA NOT MET.

(c) $y_o(n) = (-1)^n y_d(n)$ BUT $-1 = e^{j\pi}$

$$= e^{j\pi n} y_d(n)$$
$$= \cos(\pi n)\, y_d(n)$$
$$= \cos(\pi n)\left\{\cos\left[\tfrac{4\pi}{5} n\right] + \cos\left[\tfrac{2\pi}{5} n\right]\right\}$$

$$= \tfrac{1}{2}\left\{\cos\left[\tfrac{9\pi}{5} n\right] + \cos\left[\tfrac{\pi}{5} n\right] + \cos\left[\tfrac{7\pi}{5} n\right] + \cos\left[\tfrac{3\pi}{5} n\right]\right\}$$

NOTE: ABOVE STEP FROM $\cos A \cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$

AS IN PART (b), ALIASING HAS OCCURRED $\rightarrow$ $\omega_d > \pi$ ARE ALIASED BACK INTO $(-\pi, \pi)$ WITH

$$\hat{\omega}_d = |\omega_d - 2\pi|$$

GIVING: $\pm \pi/5$, $\pm 3\pi/5$

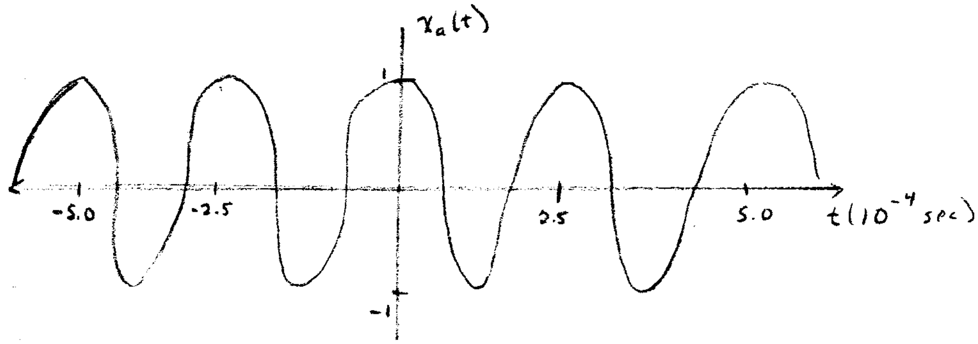
IN RANGE $0 \le \omega_d \le \pi$: $\pi/5$, $3\pi/5$

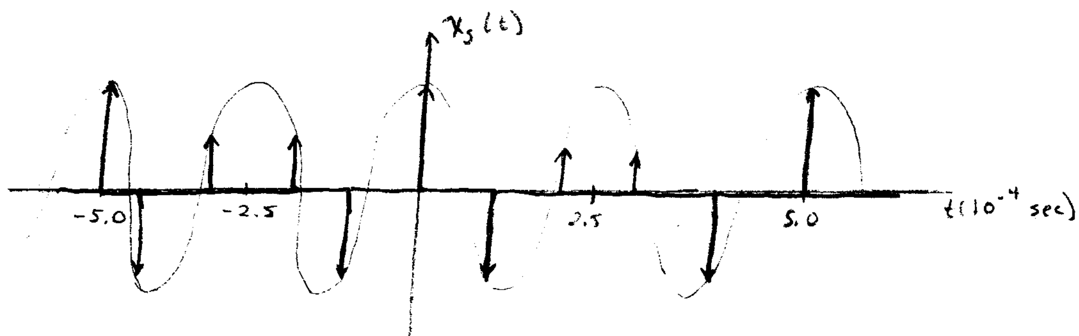
3) $x_a(t) = \cos(2\pi\, 4000\, t)$ $\quad f_s = 10$ kHz

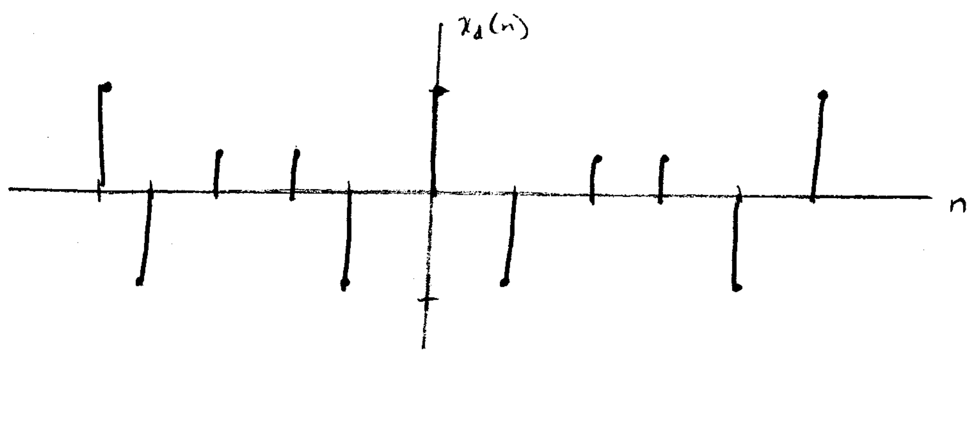
$x_s(t) = \text{comb}_T\,[x_a(t)]$

$x_d(n) = x_a(nT)$

(a)



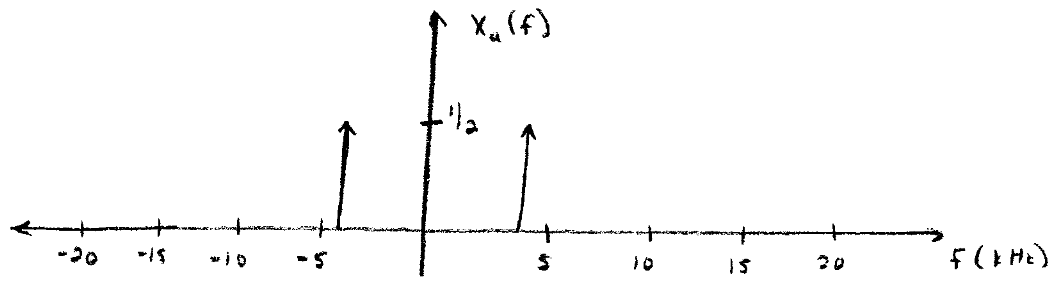


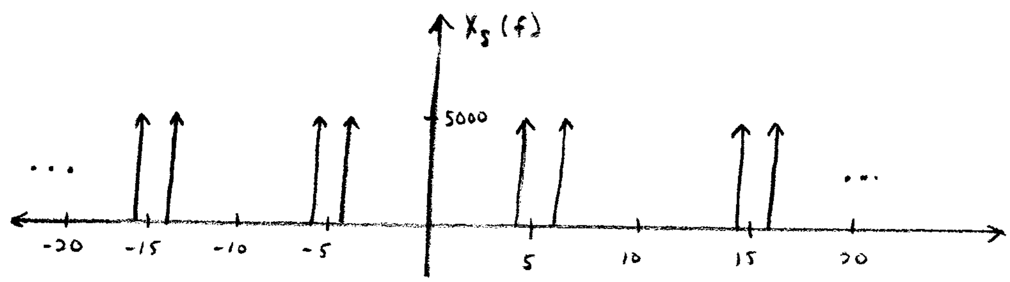


(b) $X_a(f) = \frac{1}{2}[\delta(f-4000) + \delta(f+4000)]$

$$X_s(f) = \frac{1}{T}\, rep_{\frac{1}{T}}[X_a(f)]$$

$$= 10000\, rep_{10000}\left\{\frac{1}{2}[\delta(f-4000) + \delta(f+4000)]\right\}$$





(c) $X_d(\omega) = X_s\left(\frac{\omega}{2\pi T}\right)$

$$X_s(f) = 5000\, rep_{10000}[\delta(f-4000) + \delta(f+4000)]$$

$$= 5000 \sum_{k=-\infty}^{\infty}[\delta(f-4000-10000k) + \delta(f+4000-10000k)]$$

$$X_d(\omega) = 5000 \sum_{k}\left[\delta\left(\frac{\omega}{2\pi T} - 4000 - 10000k\right) + \delta\left(\frac{\omega}{2\pi T} + 4000 - 10000k\right)\right]$$

$$= 5000 \sum_{k}\left[\delta\left(\frac{10000\,\omega}{2\pi} - 4000 - 10000k\right) + \delta\left(\frac{10000\,\omega}{2\pi} + 4000 - 10000k\right)\right]$$

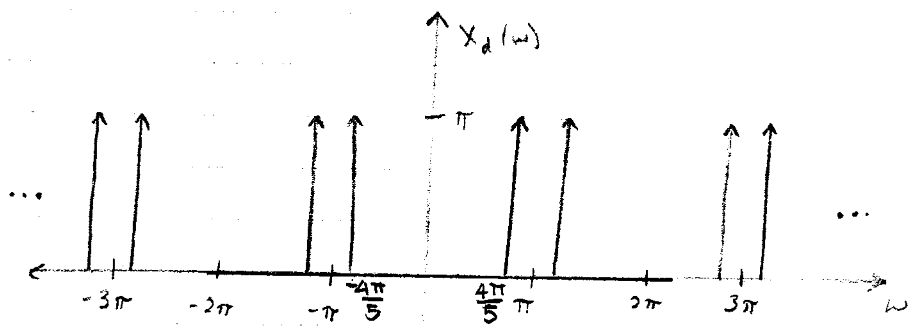
(next)

FROM HINT: $\delta(A\omega - b) = \frac{1}{|A|}\,\delta\left(\omega - \frac{b}{|A|}\right)$

$$X_d(\omega) = 5000 \sum_k \left\{ \frac{2\pi}{10000} \left[\delta\left(\omega - \frac{2\pi}{10000}(10000k + 4000)\right) + \delta\left(\omega - \frac{2\pi}{10000}(10000k - 4000)\right)\right]\right\}$$

$$= \pi \sum_k \left[\delta\left(\omega - 2\pi k - 4\pi/5\right) + \delta\left(\omega - 2\pi k + 4\pi/5\right)\right]$$

(d)



(e) $x_a(t) = \cos(2\pi\, 4000 t)$

$f_s = 10\,\text{KHz}$

$$x_d(n) = x_a\left(\frac{n}{f_s}\right) = \cos\left(2\pi \frac{4000}{10{,}000} n\right) = \cos\left(\frac{4\pi}{5} n\right)$$

- in class, we showed:

$$\cos(\omega_0 n) \xleftrightarrow{\text{DTFT}} \pi \sum_{k=-\infty}^{\infty} \left\{ \delta(\omega - \omega_0 - 2\pi k) + \delta(\omega + \omega_0 - 2\pi k) \right\}$$

Thus,

$$\cos\left(\frac{4\pi}{5} n\right) \xleftrightarrow{\text{DTFT}} \pi \sum_{k=-\infty}^{\infty} \left\{ \delta\left(\omega - \frac{4\pi}{5} - 2\pi k\right) + \delta\left(\omega + \frac{4\pi}{5} - 2\pi k\right) \right\}$$

for $|\omega| < \pi$: $\text{DTFT}\left\{\cos\left(\frac{4\pi}{5} n\right)\right\} = \pi \left\{ \delta\left(\omega - \frac{4\pi}{5}\right) + \delta\left(\omega + \frac{4\pi}{5}\right)\right\}$

SAME AS PARTS (c) & (d).

4) $E\{x(t)\} = 0$ (ZERO MEAN)

$E\{x^2(t)\} = \sigma^2$ (VARIANCE)

For the Gaussian

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$$

- desire overload probability < 0.02
- $M \approx 2^B$ levels (actually $2^B - 1$)
- assume symmetry $\Rightarrow$ swing from $-\frac{1}{2}2^B = -2^{B-1}$ to $+2^{B-1}$

$\therefore \quad P\{|x| > 2^{B-1}\Delta\} = 0.02$ (NOTE: 2^B levels at Δ each)

$\Rightarrow P\{x > 2^{B-1}\Delta\} = 0.01$ (by symmetry)

$P\{0 \le x \le 2^{B-1}\Delta\} = 0.5 - 0.01 = 0.49$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_0^{2^{B-1}\Delta} e^{-t^2/2\sigma^2}\, dt = 0.49$$

Setting $y = t/\sigma$,

$$\frac{1}{\sqrt{2\pi}} \int_0^{\frac{2^{B-1}\Delta}{\sigma}} e^{-y^2/2}\, dy = 0.49$$

From Gaussian data tables,

$$\frac{2^{B-1}\Delta}{\sigma} = 2.35$$

$$\Delta = 2.35\,\sigma \cdot 2^{1-B}$$

(next page)

Now, analyzing the noise:

$n_q(t)$ is uniformly distributed between $-\Delta/2$ and $\Delta/2$.

$$E\{n_q(t)\} = 0 \qquad \underbrace{E\{n_q^2(t)\} = \Delta^2/12}_{\text{JUST ABOUT ANY COMMUNICATIONS OR SIGNAL PROCESSING BOOK HAS THIS, ALSO EASY TO SHOW.}}$$

$$SNR = 10 \log_{10} \frac{E\{x^2(t)\}}{E\{n_q^2(t)\}} = 10 \log_{10} \frac{\sigma^2}{(2.35\sigma\, 2^{1-B})^2/12}$$

$$= 10 \log_{10} \frac{2^{2B} \cdot 12}{4 \cdot (2.35)^2} = -2.65 + 6B$$

5)

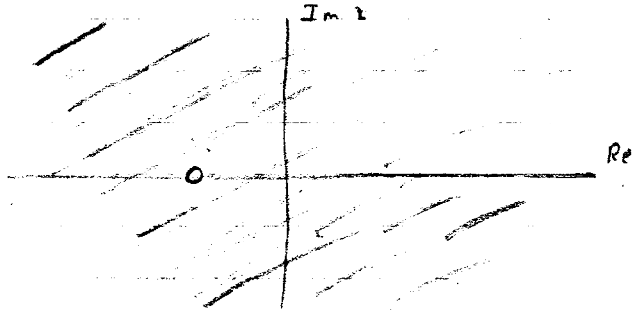
(a) $x(n) = \delta(n+2) + 2\delta(n+1) + \delta(n)$

$$X(z) = \sum_{n=0}^{2} x(n) z^{-n}$$

$$= z^2 + 2z + 1 \qquad \text{ROC: } |z| < \infty$$

POLES: ∞

ZEROES: $z^2 + 2z + 1 = (z+1)^2 \Rightarrow z = -1$



(b) $x(n) = \delta(n+1) + 2\,\delta(n) + \delta(n-1)$

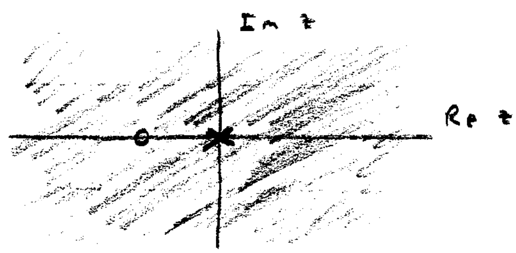
$$X(z) = \sum_{n=-1}^{1} x(n)\, z^{-n} = z + 2 + z^{-1}$$

$$= z^{-1}\left(z^2 + 2z + 1\right) = \frac{(z+1)(z+1)}{z}$$

ROC: $0 < |z| < \infty$

POLES: $0, \infty$

ZEROES: $(z+1) = 0 \Rightarrow z = -1$



(c) $x(n) = e^{j\omega_0 n} \quad -\infty < n < \infty$

$$e^{j\omega_0 n} u(n) \xleftrightarrow{ZT} \frac{1}{1 - e^{j\omega_0} z^{-1}}, \quad |z| > |e^{j\omega_0}| = 1 \Big\} \text{ outside unit circle}$$

$$e^{j\omega_0 n} u(-n-1) \xleftrightarrow{ZT} \frac{-1}{1 - e^{j\omega_0} z^{-1}}, \quad |z| < |e^{j\omega_0}| = 1 \Big\} \text{ inside unit circle}$$

ROC's do not overlap

$\Rightarrow X(z)$ does not exist

(d)

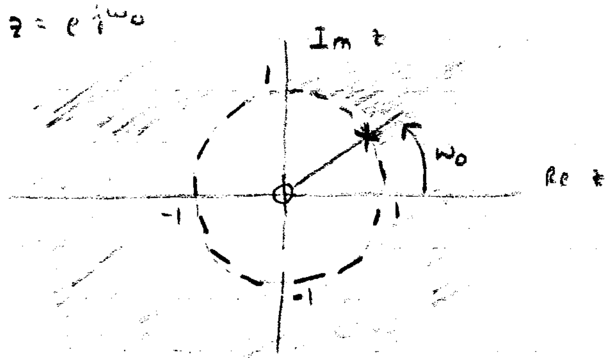
$$x(n) = e^{jw_0 n} u(n)$$

$$e^{jw_0 n} u(n) \xleftrightarrow{ZT} \frac{1}{1 - e^{jw_0} z^{-1}}, \qquad |z| > |e^{jw_0}| = 1$$

$$X(z) = \frac{z}{z - e^{jw_0}}$$

ZERO: $z = 0$

POLE: $z = e^{jw_0}$



(e) $x(n) = \sin(w_0 n)\, u(n)$

$$= \frac{1}{2j}\left(e^{jw_0 n} - e^{-jw_0 n}\right)$$

$$e^{jw_0 n} u(n) \xleftrightarrow{ZT} \frac{1}{1 - e^{jw_0} z^{-1}} \qquad |z| > |e^{jw_0}| = 1$$

$$e^{-jw_0 n} u(n) \xleftrightarrow{ZT} \frac{1}{1 - e^{-jw_0} z^{-1}} \qquad |z| > |e^{-jw_0}| = 1$$

$$X(z) = \frac{1}{2j}\left[\frac{z}{z - e^{jw_0}} - \frac{z}{z - e^{-jw_0}}\right]$$

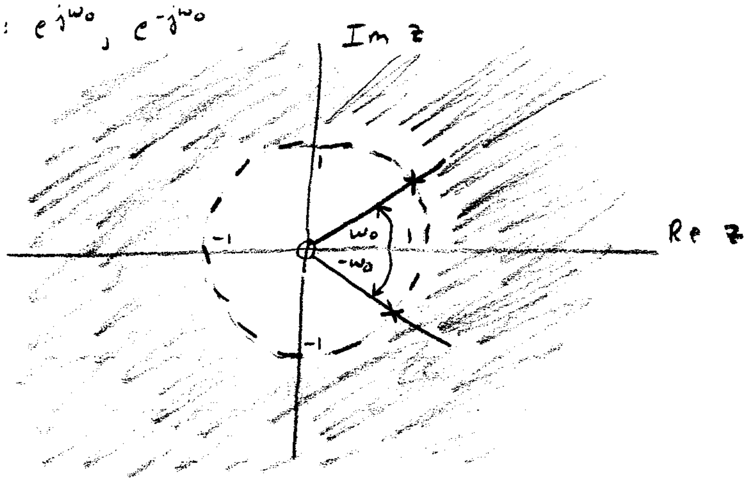
$$= \frac{1}{2j}\left[\frac{z^2 - z e^{-jw_0} - z^2 + z e^{jw_0}}{z^2 - z e^{-jw_0} - z e^{jw_0} + 1}\right]$$

$$= \frac{1}{2j}\left[\frac{z\left(e^{jw_0} - e^{-jw_0}\right)}{z^2 - 2z\cos w_0 + 1}\right] = \frac{z \sin w_0}{z^2 - 2z\cos w_0 + 1}$$

(next)

ZERO: $z=0$

POLES: $e^{j\omega_0}$, $e^{-j\omega_0}$



f)

$$x(n) = 2^n u(-n) + 4^n u(n)$$

$$4^n u(n) \xleftrightarrow{ZT} \frac{1}{1-4z^{-1}} \qquad |z| > 4$$

$$2^n u(-n) = 2^n u(-n-1) + \delta(n) \xleftrightarrow{ZT} \frac{-1}{1-2z^{-1}} + 1 \qquad |z| < 2$$

$$\{|z| < 4\} \cap \{|z| < 2\} = \phi$$

$\therefore$ NO ROC $\Rightarrow$ ZT does not exist

(g)

$$x(n) = -3^n u(-n-1) + e^{j\omega_0 n} u(n)$$

$$-3^n u(-n-1) \xleftrightarrow{ZT} \frac{1}{1-3z^{-1}} \quad |z| < 3$$

$$e^{j\omega_0 n} u(n) \xleftrightarrow{ZT} \frac{1}{1-e^{j\omega_0}z^{-1}} \quad |z| > 1$$

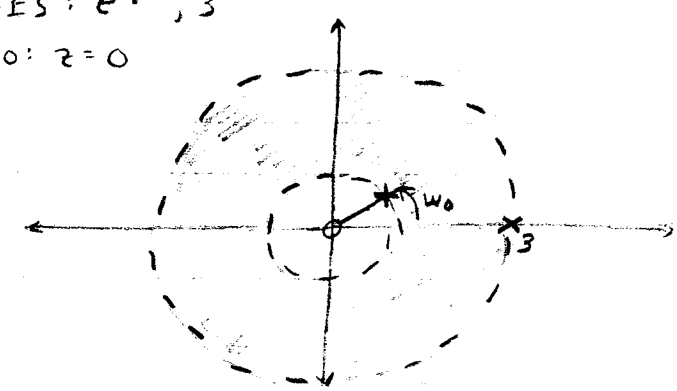
$$X(z) = \frac{z}{z-3} + \frac{z}{z-e^{j\omega_0}}$$

$$= \frac{z^2 - z e^{j\omega_0} + z^2 - 3z}{z^2 - 3z - z e^{j\omega_0} - 3e^{j\omega_0}}$$

$$= \frac{2z^2 - (e^{j\omega_0}+3)z}{z^2 - (e^{j\omega_0}+3)z - 3e^{j\omega_0}}$$

POLES: $e^{j\omega_0}, 3$

ZERO: $z = 0$



(h) USING PROPERTY $n x(n) \xleftrightarrow{ZT} -z \frac{dX(z)}{dz}$

$$u[n] \xleftrightarrow{ZT} \frac{1}{1-z^{-1}}$$

$$n\, u[n] \xleftrightarrow{ZT} -z \frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right) = -z\left(\frac{-z^{-2}}{(1-z^{-1})^2}\right)$$

$$= \frac{z^{-1}}{(1-z^{-1})^2} \qquad \text{ROC: } |z| > 1$$

(next)

ZERO: $z = 0$

POLES: 1