Solutions to Assignment 2

- Problem 1
- first, note: $\text{DTFT}\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) e^{-j\omega n}$

$n' = -n$ $$= \sum_{n'=-\infty}^{\infty} x(n') e^{+j\omega n'}$$

$$= X^*(e^{j\omega}) \quad \text{if } x(n) \text{ is real}$$

$$x(-n) \xleftrightarrow{\text{DTFT}} X^*(e^{j\omega})$$

(a) $$x_e(n) = \frac{1}{2}\{x(n) + x(-n)\}$$

taking DTFT of both sides $$X_e(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{j\omega})\}$$
$$= \text{Re}\{X(e^{j\omega})\} \quad \#$$

similarly, $$x_o(n) = \frac{1}{2}\{x(n) - x(-n)\}$$
$$X_o(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{j\omega})\}$$
$$= j\,\text{Im}\{X(e^{j\omega})\}$$

(b) $$u_e(n) = \frac{1}{2}\{u(n) + u(-n)\}$$ where: $u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$
$$= \frac{1}{2}\delta(n) + \frac{1}{2}$$

taking DTFT of both sides $$U_e(e^{j\omega}) = \frac{1}{2} + \frac{1}{2} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$
$$= \frac{1}{2} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

- the odd part of $u(n)$ is a little harder

Prob. 1 (cont.), part (b)

$$u_o(n) = \frac{1}{2}\left\{u(n) - u(-n)\right\} = \begin{cases} 1/2, & n>0 \\ 0, & n=0 \\ -1/2, & n<0 \end{cases}$$

- to compute DTFT of this, define:

$$\text{rect}_N(n) = u(n-1) - u(n-(N+1))$$

- then:

$$u_o(n) = \frac{1}{2}\lim_{N\to\infty}\left\{\text{rect}_N(n) - \text{rect}_N(-n)\right\}$$

- and

$$R_N(e^{j\omega}) = \text{DTFT}\left\{\text{rect}_N(n)\right\}$$

- then:

$$U_o(e^{j\omega}) = \lim_{N\to\infty} j\,\text{Im}\left\{R_N(e^{j\omega})\right\}$$

- now:

$$R_N(e^{j\omega}) = \sum_{n=1}^{N} e^{-j\omega n} = e^{-j\omega}\sum_{n'=0}^{N-1} e^{-j\omega n'}$$

$$= e^{-j\omega}\left\{\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right\} = e^{-j\frac{\omega}{2}(N+1)}\frac{\sin(N\frac{\omega}{2})}{\sin(\frac{\omega}{2})}$$

- Hence,

$$U_o(e^{j\omega}) = \lim_{N\to\infty}\frac{1}{2}\left\{e^{-j\frac{\omega}{2}(N+1)} - e^{+j\frac{\omega}{2}(N+1)}\right\}\frac{\sin(N\frac{\omega}{2})}{\sin(\frac{\omega}{2})}$$

Remember:
$\text{Im}\{A\} = \frac{1}{2j}\{A - A^*\}$

$$= \lim_{N\to\infty} -j\,\frac{\sin((N+1)\frac{\omega}{2})\sin(N\frac{\omega}{2})}{\sin(\frac{\omega}{2})}$$

$$= -j\lim_{N\to\infty}\frac{1}{2}\frac{\cos(\frac{\omega}{2})}{\sin(\frac{\omega}{2})} + \underbrace{j\lim_{N\to\infty}\frac{1}{2}\frac{\cos((2N+1)\frac{\omega}{2})}{\sin(\frac{\omega}{2})}}_{=\varnothing}$$

$$= -\frac{j}{2}\cot\left(\frac{\omega}{2}\right)$$

Prob. 1, part (b) (cont.)

- by linearity:

$$U(e^{j\omega}) = U_e(e^{j\omega}) + U_o(e^{j\omega})$$

$$U(e^{j\omega}) = \frac{1}{2} - \frac{j}{2}\cot\left(\frac{\omega}{2}\right) + \pi\sum_{k=-\infty}^{\infty}\delta(\omega - 2\pi k)$$

- (c) We are given:

$$U(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} + \pi\sum_{k=-\infty}^{\infty}\delta(\omega - 2\pi k)$$

- comparing with the above, we nee' to show:

$$\underbrace{\frac{1}{1-e^{-j\omega}}}_{\text{let's work on this}} = \frac{1}{2} - \frac{j}{2}\cot\left(\frac{\omega}{2}\right)$$

$$\frac{1}{1-e^{-j\omega}} = \frac{e^{j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} = \frac{e^{j\frac{\omega}{2}}}{2j\sin\left(\frac{\omega}{2}\right)} = \frac{\cos\left(\frac{\omega}{2}\right) + j\sin\left(\frac{\omega}{2}\right)}{2j\sin\left(\frac{\omega}{2}\right)}$$

$$= \frac{1}{2} + \frac{1}{2j}\cot\left(\frac{\omega}{2}\right) = \frac{1}{2} - \frac{1}{2}j\cot\left(\frac{\omega}{2}\right)$$

- Q.E.D.

Problem 2. $y(n) = 2\cos\omega_o\, y(n-1) - y(n-2) + A\sin\omega_o\, x(n)$

- taking the DTFT of both sides

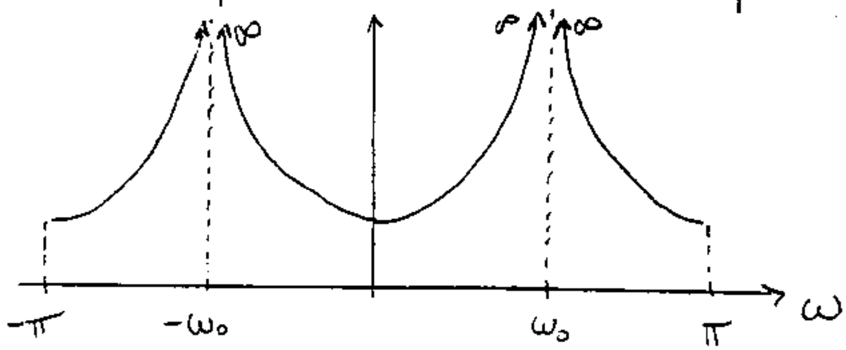
$$Y(e^{j\omega})\left\{1 - 2\cos\omega_o\, e^{-j\omega} + e^{-j2\omega}\right\} = A\sin\omega_o\, X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{A\sin\omega_o}{1 - 2\cos\omega_o\, e^{-j\omega} + e^{-j2\omega}}$$

$$= \frac{A\sin\omega_o}{(e^{-j\omega} - e^{-j\omega_o})(e^{-j\omega} - e^{j\omega_o})}$$

Prob. 2, part (a) (cont.)

$$|H(e^{j\omega})| = \frac{A\,|\sin\omega_0|}{|e^{-j\omega} - e^{-j\omega_0}|\,|e^{-j\omega} - e^{j\omega_0}|}$$



drawn for the case of $\omega_0 = \frac{\pi}{2}$

- $H(e^{j\omega})\Big|_{\omega=\omega_0} = \infty$

(b) We do this by showing that

$$\sin(\omega_0 n)u(n) \overset{DTFT}{\longleftrightarrow} \frac{\sin\omega_0\, e^{-j\omega}}{1 - 2\cos\omega_0\, e^{-j\omega} + e^{-j2\omega}}$$

- first, note:

$$\frac{e^{-j\omega}\sin\omega_0}{1 - 2\cos\omega_0 e^{-j\omega} + e^{-j2\omega}} = \frac{\sin\omega_0\, e^{-j\omega}}{(e^{-j\omega} - e^{-j\omega_0})(e^{-j\omega} - e^{+j\omega_0})} = \frac{A_1}{e^{-j\omega} - e^{-j\omega_0}} + \frac{A_2}{e^{-j\omega} - e^{+j\omega_0}}$$

$$A_1 = \frac{\sin\omega_0\, e^{-j\omega}}{e^{-j\omega} - e^{j\omega_0}}\Bigg|_{\omega=\omega_0} = \frac{\sin\omega_0\, e^{-j\omega_0}}{e^{-j\omega_0} - e^{j\omega_0}} = \frac{j}{2}\, e^{-j\omega_0} = A_2^*$$

- now, invoking the result of problem 1, the modulation property of the DTFT dictates

$$e^{j\omega_0 n}u(n) \overset{DTFT}{\longleftrightarrow} \frac{1}{1 - e^{-j(\omega-\omega_0)}} + \pi\sum_{k=-\infty}^{\infty}\delta(\omega - \omega_0 - 2\pi k)$$

$$= \frac{e^{-j\omega_0}}{e^{-j\omega_0} - e^{-j\omega}} + \pi\sum_{k=-\infty}^{\infty}\delta(\omega - \omega_0 - 2\pi k)$$

$$= \frac{-e^{-j\omega_0}}{e^{-j\omega} - e^{-j\omega_0}} + \pi\sum_{k=-\infty}^{\infty}\delta(\omega - \omega_0 - 2\pi k)$$

Prob 2, part (b) (cont.)

· similarly,

$$e^{-j\omega_0 n} u(n) \overset{DTFT}{\longleftrightarrow} \frac{e^{j\omega_0}}{e^{-j\omega} - e^{j\omega_0}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi k)$$

• now, since $\sin(\omega_0 n) u(n) = \frac{1}{2j}\left\{ e^{j\omega_0 n} - e^{-j\omega_0 n} \right\} u(n)$

$$\sin(\omega_0 n) u(n) \overset{DTFT}{\longleftrightarrow} \frac{\frac{j}{2} e^{-j\omega_0}}{e^{-j\omega} - e^{-j\omega_0}} + \frac{-\frac{j}{2} e^{j\omega_0}}{e^{-j\omega} - e^{j\omega_0}}$$

$$+ \frac{\pi}{2j}\left\{ \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) - \sum_{k=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi k) \right\}$$

• this agrees with the result written previously except for a set of ω of measure zero (at discrete points $\{\pm\omega_0 \pm 2\pi k, \ k]_{-\infty}^{\infty}\}$)

• verification: let $x(n) = \delta(n)$ such that $y(n) = h(n)$

$$h(n) = 2\cos\omega_0 \, h(n-1) - h(n-2) + A \sin\omega_0 \, \delta(n)$$

· since system is causal, $h(-1) = h(-2) = 0$

$n=0$: $h(0) = A \sin\omega_0 \, \delta(0) = A\sin\omega_0$

$n=1$: $h(1) = 2\cos\omega_0 \underbrace{A\sin\omega_0}_{h(1)} = A\sin 2\omega_0$

$n=2$: $h(2) = 2\cos\omega_0 A \sin 2\omega_0 - A\sin\omega_0$

$$= A\sin\omega_0 \left\{ 4\cos^2\omega_0 - 1 \right\} \qquad \cos^2\omega_0 = 1 - \sin^2\omega_0$$

$$= A\sin\omega_0\left\{3 - 4\sin^2\omega_0\right\}$$

$$= A\left\{3\sin\omega_0 - 4\sin^3\omega_0\right\} \qquad \text{see trig. tables}$$

$$= A\sin(3\omega_0)$$

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Problem 3

(a) $$V(e^{j\omega}) = \sum_{n=-\infty}^{\infty} v(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, y(n)\, e^{-j\omega n}$$

Substitute inverse DTFT for $y(n)$

$$= \sum_{n=-\infty}^{\infty} x(n)\, \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\mu}) e^{j\mu n}\, d\mu \; e^{-j\omega n}$$

interchange order of summation and integration

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\mu}) \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\mu)n}\, d\mu$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\mu})\, X(e^{j(\omega-\mu)})\, d\mu$$

periodic convolution is commutative

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\mu})\, Y(e^{j(\omega-\mu)})\, d\mu = V(e^{j\omega})$$

$$= X(e^{j\omega}) \circledast Y(e^{j\omega}) \quad \Rightarrow \circledast \text{ denotes periodic convolution}$$

(b) $$x(n) = \frac{\sin^2(\frac{\pi}{4}n)}{(\pi n)^2} = \frac{\sin(\frac{\pi}{4}n)}{\pi n}\;\frac{\sin(\frac{\pi}{4}n)}{\pi n}$$

• note:

$$\frac{\sin(\omega_o n)}{\pi n} \xleftrightarrow{\text{DTFT}} \text{rect}\left(\frac{\omega}{2\omega_o}\right) \quad \text{for } -\pi < |\omega| < \pi$$

• proof:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \text{rect}\left(\frac{\omega}{2\omega_o}\right) e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_o}^{\omega_o} e^{j\omega n}\, d\omega = \left.\frac{e^{j\omega n}}{j2\pi n}\right]_{-\omega_o}^{\omega_o}$$

$$= \frac{\sin \omega_o n}{\pi n}$$

• thus:

$$\frac{\sin(\frac{\pi}{4}n)}{\pi n} \xleftrightarrow{\text{DTFT}} \underbrace{\text{rect}\left(\frac{\omega}{\pi/2}\right) \quad \text{for } -\pi < |\omega| < \pi}_{\text{remember: DTFT is periodic with period } 2\pi}$$

Prob. 3, part (b) (cont.):

- let ⊛ denote periodic convolution as in part (a)

$$\frac{\sin^2\left(\frac{\pi}{4}n\right)}{\pi^2 n^2} \xleftrightarrow{\text{DTFT}} \underbrace{\overbrace{\mathrm{rect}\left(\frac{\omega}{\pi/2}\right) \circledast \mathrm{rect}\left(\frac{\omega}{\pi/2}\right)}^{\text{remember: DTFT is periodic with period } 2\pi}}_{=\frac{\pi}{2}\Lambda\left(\frac{\omega}{\pi/2}\right) \text{ for } -\pi < |\omega| < \pi} \quad \text{for } -\pi < |\omega| < \pi$$

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Problem 4

(a) for $y(n)$, which is complex-valued in general

$$\mathrm{DTFT}\{y^*(-n)\} = \sum_{n=-\infty}^{\infty} y^*(-n)\, e^{-j\omega n}$$

change of variables: $n' = -n$

$$= \sum_{n'=-\infty}^{\infty} y^*(n')\, e^{j\omega n'} = Y^*(e^{j\omega})$$

- since, we also have:

$$x(n) * y(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})\, Y(e^{j\omega})$$

- from the above observation, it follows that

$$x(n) * y^*(-n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})\, Y^*(e^{j\omega})$$

- (b) the result above implies that

$$\sum_{k=-\infty}^{\infty} x(k)\, y_r(n-k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, Y^*(e^{j\omega})\, e^{j\omega n}\, d\omega$$

where: $y_r(n) = y^*(-n)$

Prob. 4, part (b) (cont.)

- since $y_r(\xi) = y^*(-\xi) \Rightarrow y_r(n-k) = y^*(-n+k)$,
- substituting:

$$\sum_{k=-\infty}^{\infty} x(k)\, y^*(-n+k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, Y^*(e^{j\omega})\, e^{j\omega n}\, d\omega$$

- works for any $n \Rightarrow$ in particular $n=0$:

$$\sum_{k=-\infty}^{\infty} x(k)\, y^*(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, Y^*(e^{j\omega})\, d\omega$$

- k is just a dummy counting variable:

$$\sum_{n=-\infty}^{\infty} x(n)\, y^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, Y^*(e^{j\omega})\, d\omega$$

(c) using the result above:

$$\left(\frac{1}{2}\right)\left(\frac{1}{5}\right)\sum_{n=-\infty}^{\infty} \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}\,\frac{\sin\left(\frac{\pi}{6}n\right)}{\pi n} = \left(\frac{1}{10}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi} \text{rect}\left(\frac{\omega}{\pi/2}\right)\text{rect}\left(\frac{\omega}{\pi/3}\right) d\omega$$

$$= \left(\frac{1}{10}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi} \text{rect}\left(\frac{\omega}{\pi/3}\right) d\omega$$

$$= \left(\frac{1}{10}\right)\frac{1}{2\pi}\underbrace{\left(\frac{\pi}{3}\right)}_{\text{area under } \text{rect}\left(\frac{\omega}{\pi/3}\right)} = \frac{1}{60}$$

- thus:

$$\sum_{n=-\infty}^{\infty} \frac{\sin\left(\frac{\pi}{4}n\right)}{2\pi n}\,\frac{\sin\left(\frac{\pi}{6}n\right)}{5\pi n} = \frac{1}{60}$$

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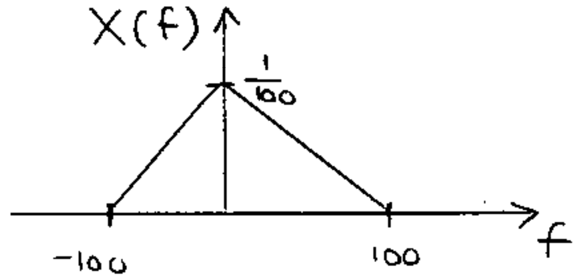
Problem 5

(a) • first, note: $\mathrm{sinc}(t) \xleftrightarrow{CTFT} \mathrm{rect}(f)$

thus: $\mathrm{sinc}(100t) \xleftrightarrow{CTFT} \frac{1}{100}\,\mathrm{rect}\left(\frac{f}{100}\right)$ } Scaling Property

and $\mathrm{sinc}^2(100t) \xleftrightarrow{CTFT} \frac{1}{100^2}\,\mathrm{rect}\left(\frac{f}{100}\right) * \mathrm{rect}\left(\frac{f}{100}\right)$

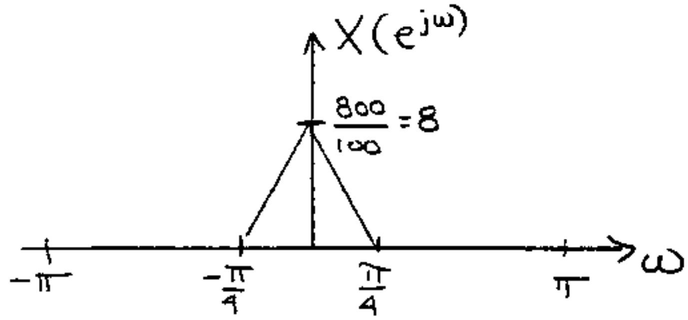
$\mathrm{sinc}^2(100t) \xleftrightarrow{CTFT} \frac{1}{100}\,\Lambda\left(\frac{f}{100}\right)$



• highest frequency in $x(t) = \mathrm{sinc}^2(100t)$ is $W = 100$ Hz

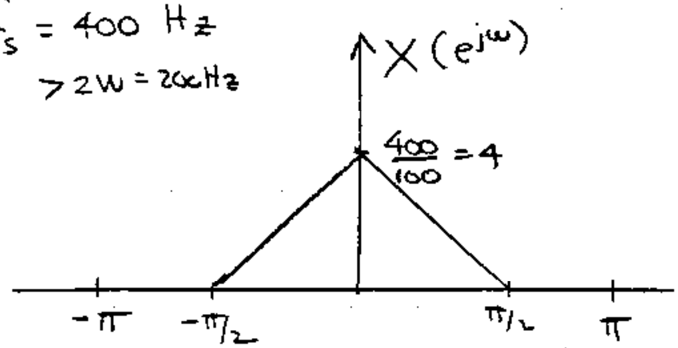
• $x(n) = x_a(nT) = x_a\left(\frac{n}{f_s}\right)$

• (i) $f_s = 800$ Hz $> 2W = 200$ Hz



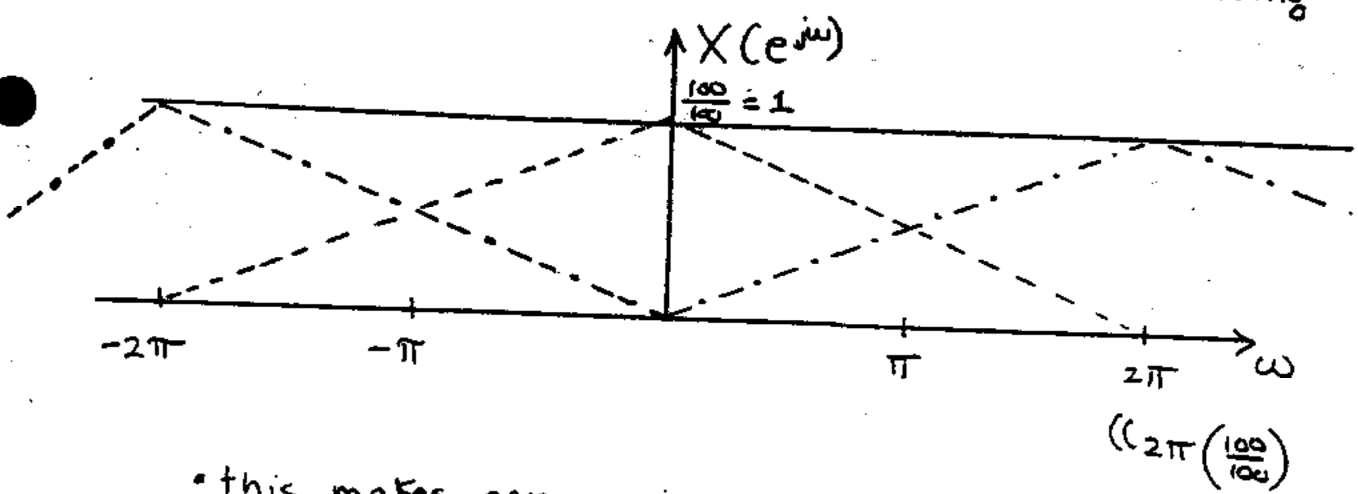
$\frac{\pi}{4} = 2\pi\left(\frac{100}{800}\right)$

(ii) $f_s = 400$ Hz $> 2W = 200$ Hz



$\frac{\pi}{2} = 2\pi\left(\frac{100}{400}\right)$

(iii) $f_s = 100 < 2W = 200$ Hz $\Rightarrow$ aliasing



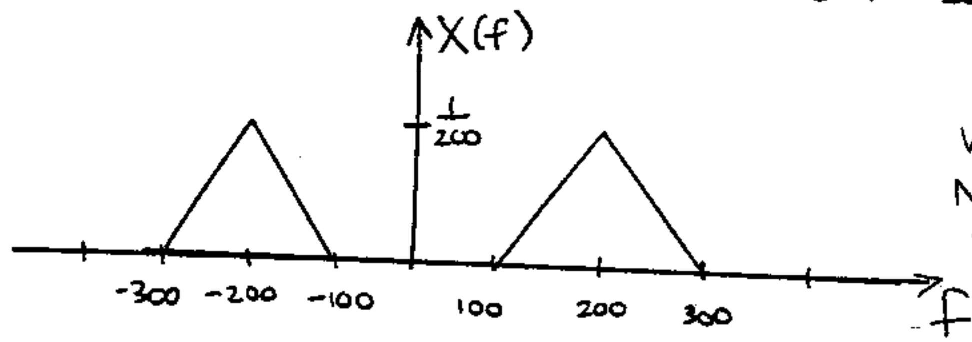
• this makes sense since

$$x(n) = x_a(nT) = \operatorname{sinc}^2\left(100\frac{n}{100}\right) = \operatorname{sinc}^2(n) = \delta(n)$$ in this case.

(b) $x_a(t) = \operatorname{sinc}^2(100t)\cos(2\pi\, 200t)$

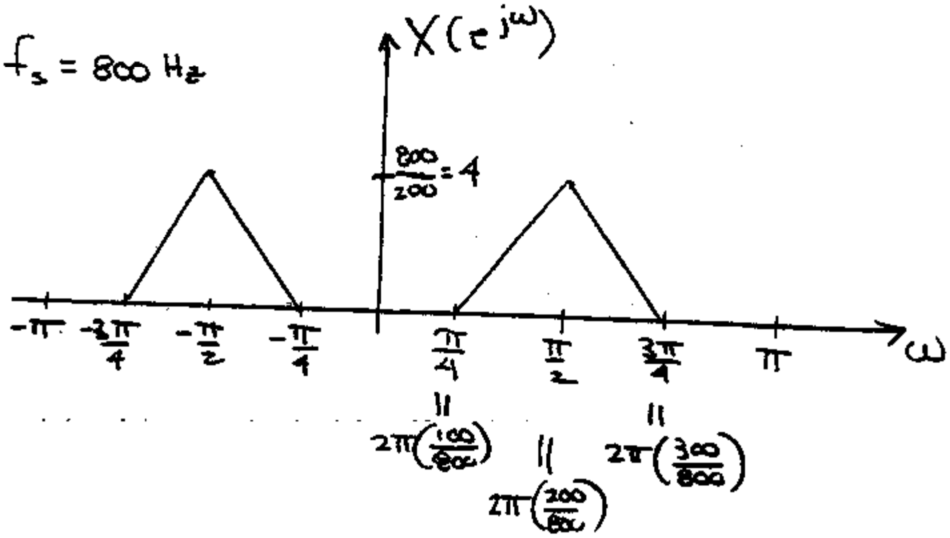
• invoking the modulation property:

$$\operatorname{sinc}^2(100t)\cos(2\pi\, 200t) \overset{CTFT}{\longleftrightarrow} \frac{1}{200}\Lambda\left(\frac{f-200}{100}\right) + \frac{1}{200}\Lambda\left(\frac{f+200}{100}\right)$$



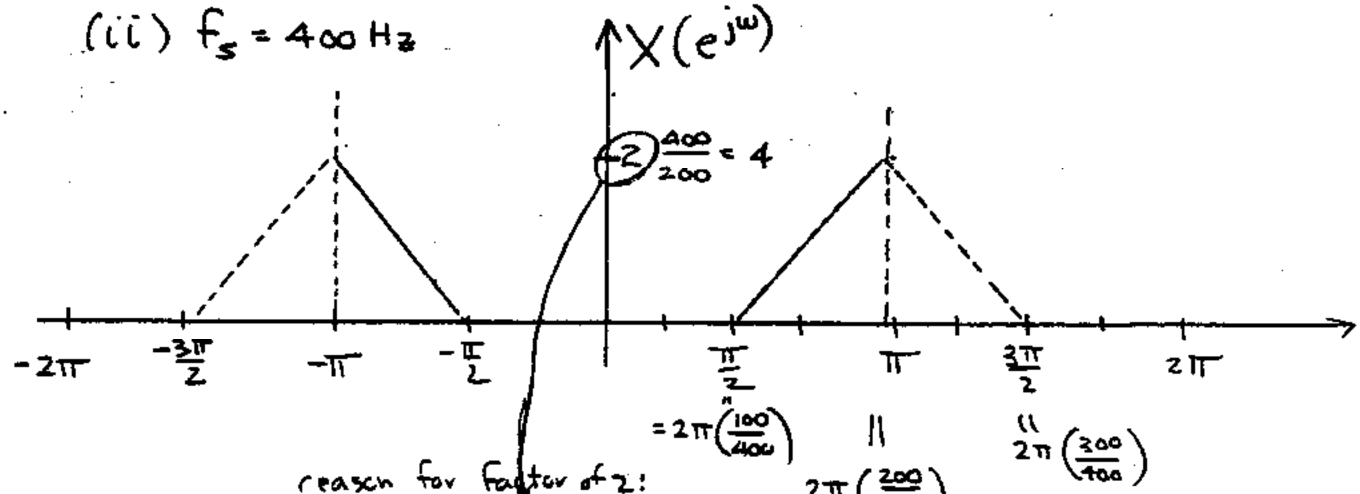
$W = 300$ Hz
Nyquist rate:
$2W = 600$ Hz

(i) $f_s = 800$ Hz



Problem 5. part (b) (cont.)

(ii) $f_s = 400$ Hz



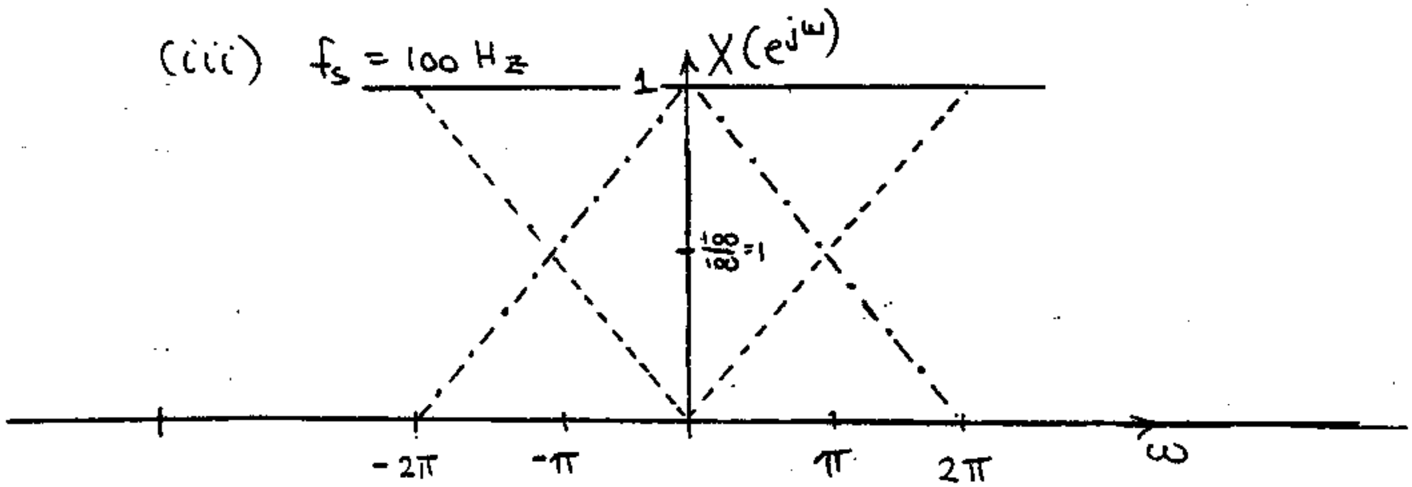
reason for factor of 2:

$$\sum_{n=-\infty}^{\infty} X_a(f - nf_s)$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} X_a\left(\frac{f_s}{2\pi}(\omega - n2\pi)\right)}$$

lower triangle of spectrum replica centered at $\omega = 2\pi$ overlaps perfectly with upper triangle of spectrum replica centered at $\omega = 0$

(iii) $f_s = 100$ Hz



· again, this makes sense since

$$x(n) = x_a\left(\frac{n}{100}\right) = \text{sinc}^2\left(100\frac{n}{100}\right)\cos\left(\frac{2\pi(200)n}{100}\right)$$

$$= \text{sinc}^2(n)\cdot\cos(4\pi n) = \delta(n)$$

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