EE438 Homework # 1
Spring 2001

1 a) i)

ii) right sided ($x(t) \neq 0$ only for $t \geq t_0 = 0$)

iii) Causal ($t_0 \geq 0$)

iv)
$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} t^2 e^{-t} dt$$

Integrate by parts:

Let $u = t^2 \Rightarrow du = 2t\,dt$

Let $dv = e^{-t}dt \Rightarrow v = -e^{-t}$

$$E_x = uv\Big|_0^{\infty} - \int_0^{\infty} v\,du$$

$$= \cancel{t^2(-e^{-t})\Big|_0^{\infty}} \to 0 - \int_0^{\infty}(-e^{-t})(2t\,dt) = \int_0^{\infty} 2t e^{-t} dt$$

Integrate by parts again.

Let $u = 2t \Rightarrow du = 2dt$

$dv = e^{-t}dt \Rightarrow v = -e^{-t}$

$$E_x = (2t)(-e^{-t})\Big|_0^\infty - \int_0^\infty (-e^{-t})\,2\,dt = \int_0^\infty 2e^{-t}\,dt$$

$$= -2e^{-t}\Big|_0^\infty = \underline{\underline{2}}$$

$P_x = \underline{\underline{0}}$ because $x(t)$ is an energy signal

$x_{rms} = \sqrt{P_x} = \underline{\underline{0}}$

$M_x = \max_t |x(t)|$. In our case $x(t) > 0 \ \forall t$, so

Solve for max: $\frac{d}{dt}x(t) = 0$

$$\frac{d}{dt} te^{-t/2} = e^{-t/2} + te^{-t/2}\left(-\tfrac{1}{2}\right) = 0$$

$$\Rightarrow t = 2e^{-1} \approx .736$$

$$\therefore M_x = 2e^{-1} \approx 0.736$$

$$A_x = \int x(t)\,dt = \int_0^\infty te^{-t/2}\,dt$$

Solve using integration by parts:

let $u = t \Rightarrow du = dt$

$dv = e^{-t/2}dt \Rightarrow v = -2e^{-t/2}$

$$A_x = uv - \int v\,du$$

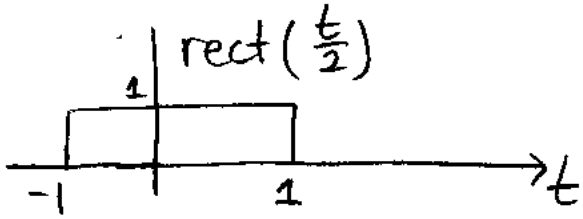
$$= t(-2e^{-t/2})\Big|_0^\infty - \int_0^\infty (-2e^{-t/2})\,dt$$

$$= \int_0^\infty 2e^{-t/2}\,dt = -4e^{-t/2}\Big|_0^\infty = \underline{\underline{4}}$$

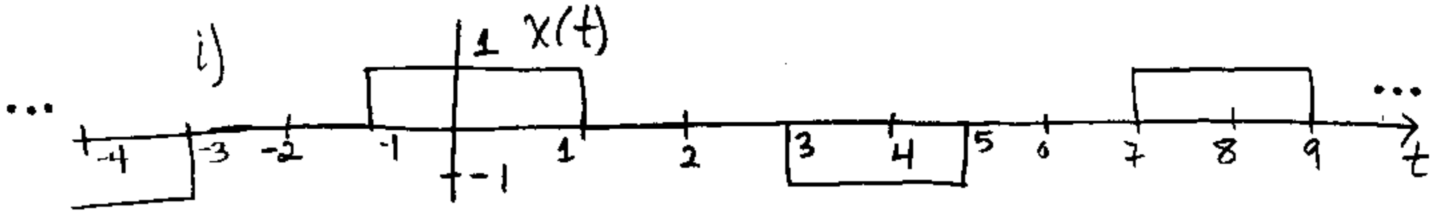
$$x_{ave} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt = \lim_{T\to\infty} \frac{1}{2T}\int_{0}^{T} t e^{-t/2}\,dt$$

$$= 0$$

b) $x(t) = \sum_k (-1)^k \operatorname{rect}\left(\frac{t}{2} - 2k\right) = \sum_k (-1)^k \operatorname{rect}\left(\frac{t-4k}{2}\right)$



i)



ii) two sided

iii) neither causal nor anticausal

iv) $E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} 1 \cdot dt = \infty$

$$P_x = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt$$

$$= \lim_{\substack{T\to\infty \\ T \text{ a multiple of } 4}} \frac{1}{2T}\int_{-T/2}^{T/2} 1 \cdot dt = \lim_{T\to\infty} \frac{T}{2T} = \underline{\underline{\frac{1}{2}}}$$

$$x_{rms} = \sqrt{P_x} = \frac{\sqrt{2}}{2}$$

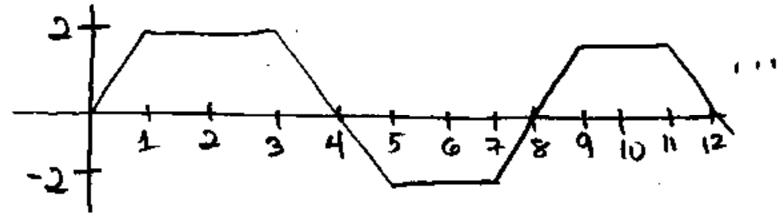
$$M_x = \max_t |x(t)| = \underline{\underline{1}}$$

$$A_x = \int_{-\infty}^{\infty} x(t)\,dt \Rightarrow \text{Undefined}$$

$$x_{ave} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} g(T) \quad \text{where } g(T) = \int_{-T}^{T} x(t)dt \text{ for } T \geq 0$$

$g(T) =$



$g(T)$ is between -2 and 2

So $\lim_{T\to\infty} \frac{1}{2T} g(T) = 0.$

So $x_{ave} = \underline{\underline{0}}$

c) $x[n] = e^{(j-1)\pi n/2} = e^{j\frac{\pi}{2}n} e^{-\frac{\pi}{2}n}$

$x[n]$ is a complex-valued sequence, so to sketch it you'll need to plot the magnitude and phase.

Tools we will use: $|AB| = |A|\,|B|$

$$\angle\frac{AB}{C} = \angle A + \angle B - \angle C$$

$$|x[n]| = \left|e^{j\frac{\pi}{2}n} e^{-\frac{\pi}{2}n}\right| = \left|e^{j\frac{\pi}{2}n}\right| \cdot \left|e^{-\frac{\pi}{2}n}\right|$$

$$= 1 \cdot e^{-\frac{\pi}{2}n}$$

magnitude of a complex exponential is 1.

Positive for all n.

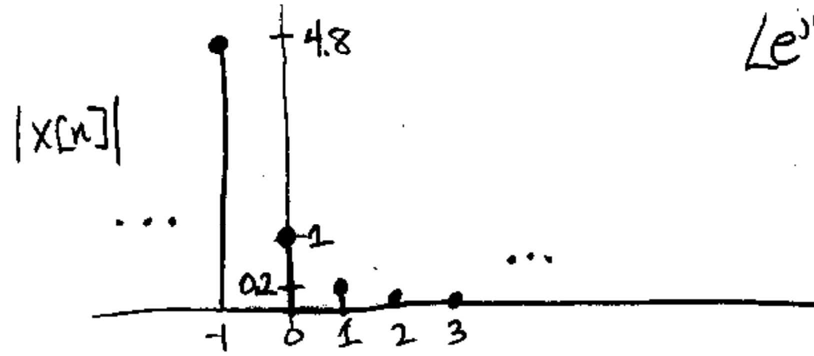
$$\angle x[n] = \angle e^{j\frac{\pi}{2}n} e^{-\frac{\pi}{2}n} = \angle e^{j\frac{\pi}{2}n} + \angle e^{-\frac{\pi}{2}n}$$

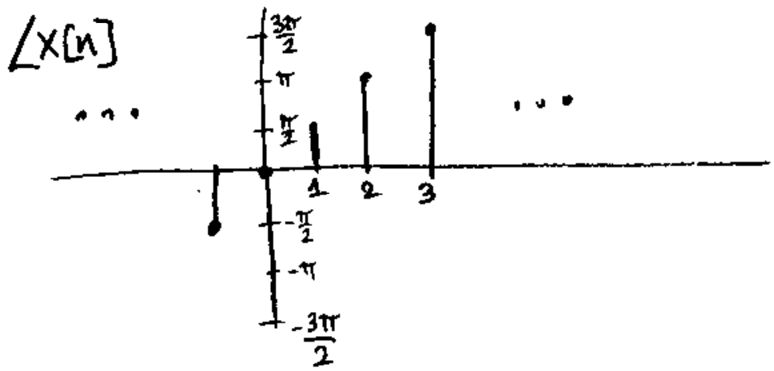
$$= \frac{\pi}{2}n + 0$$

$$= \frac{\pi}{2}n$$

This is a positive real number for all n. So its phase is zero

$$\angle e^{j\omega_0} = \omega_0$$





ii) two sided

iii) neither

iv) $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]x^*[n]$

$$= \sum_{n=-\infty}^{\infty} \cancel{e^{j\frac{\pi}{2}n}} e^{-\frac{\pi}{2}n} \cancel{e^{-j\frac{\pi}{2}n}} e^{-\frac{\pi}{2}n} = \sum_{n=-\infty}^{\infty} e^{-\pi n} = \infty$$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^{-\pi n}$$

We want to use the formula

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

So first do a change of variables.

Let $m = n+N$. Then $n = m-N$. And, if $n=N$, $m = 2N$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=0}^{2N} e^{-\pi(m-N)}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} e^{\pi N} \sum_{m=0}^{2N} (e^{-\pi})^m$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} e^{\pi N} \frac{1-(e^{-\pi})^{2N+1}}{1-(e^{-\pi})}$$

$$= \frac{1}{1-e^{-\pi}} \lim_{N\to\infty} \frac{1}{2N+1} \left(e^{\pi N} - e^{\pi(N-2N-1)}\right)$$

$$= \frac{1}{1-e^{-\pi}} \lim_{N\to\infty} \frac{1}{2N+1} \left(e^{\pi N} - e^{-\pi N} e^{-\pi}\right)$$

$$= \frac{1}{1-e^{-\pi}} \left(\lim_{N\to\infty} \frac{e^{\pi N}}{2N+1} - \lim_{N\to\infty} \frac{e^{-\pi N} e^{-\pi}}{2N+1} \right)$$

$$= \frac{1}{1-e^{-\pi}} (\infty - 0) = \underline{\underline{\infty}}$$

$$X_{rms} = \sqrt{P_x} = \infty$$

$$M_x = \max_n |x(n)| = \max_n e^{-\frac{\pi}{2}n} = \infty$$

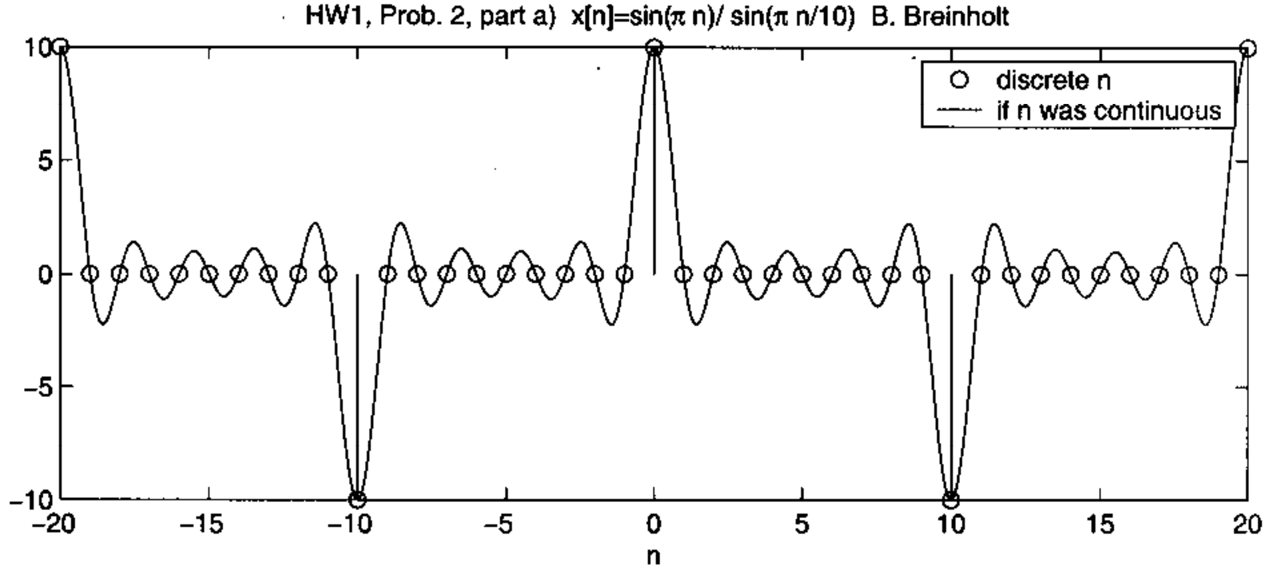
$$A_x = \sum_n x[n] = \sum_{n=-\infty}^{\infty} e^{(j-1)\frac{\pi}{2}n} = \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{2}n} \left(\cos\tfrac{\pi}{2}n + j\sin\tfrac{\pi}{2}n\right)$$

A_x is undefined
x_{ave} is undefined

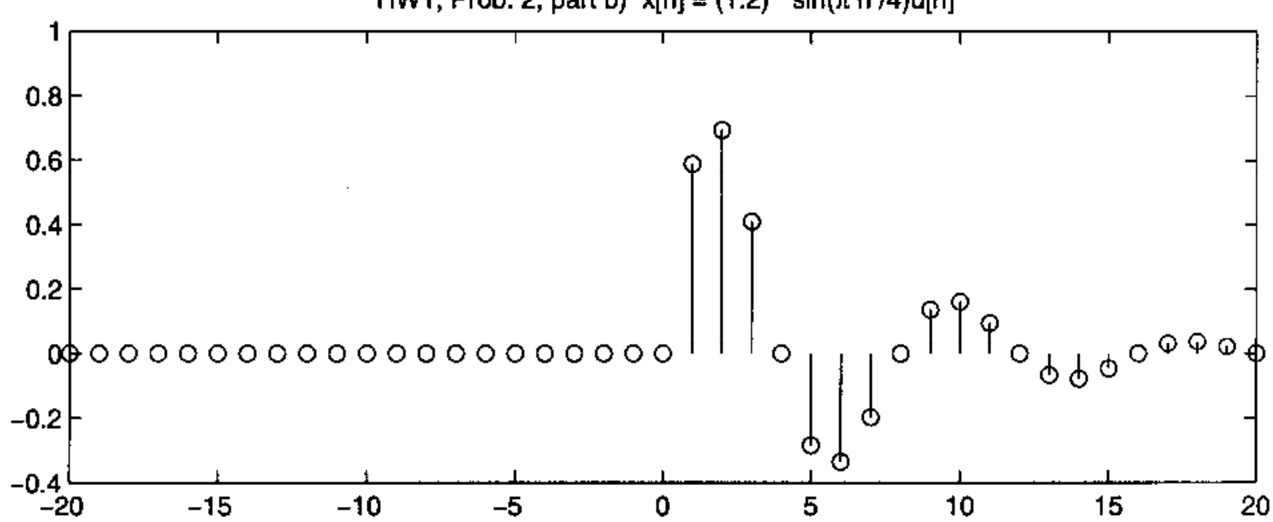
2) a) $E_x = \infty$
$P_x = 10$
$x_{rms} = 3.1623$
$M_x = 10$
$A_x = \infty$
$x_{ave} = 1$

b) $E_x = 1.2990$
$P_x = 0$
$x_{rms} = 0$
$M_x = 0.6944$
$A_x = 1.1421$
$x_{ave} = 0$

HW1, Prob. 2, part a) $x[n]=\sin(\pi n)/\sin(\pi n/10)$ B. Breinholt



HW1, Prob. 2, part b) $x[n] = (1.2)^{-n}\sin(\pi n/4)u[n]$



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% EE438 Spring 2001
% Homework Set 1, problem 2
% Brad Breinholt

% part a)

n1 = (-20:.01:20) + eps;
x1= sin(pi*n1)./sin(pi*n1 / 10);

n = -20:20;
x = sin(pi*n)./sin(pi*n/10);
x([1 11 21 31 41]) = [10 -10 10 -10 10];
                    % These are places where the quotent is undefined.
                       % They correspond to n = -20, n=-10, n=0, n=10, n=20
                       % To find the true value, replace n with the continuous
                       % variable t.  Then use L'Hopital's rule at values of t
                       % where x(t) = 0/0.  Finally go back to the discrete
                       % variable case by letting n = t.
figure
subplot(3,1,2)
stem(n,x); hold on;
plot(n1,x1,'y'); hold off
xlabel('n')
legend('discrete n', 'if n was continuous')

% Ex is infinite.
% Px is average energy.  Average over one cycle.
Px = sum(x(1:10).^2/10)
xrms = sqrt(Px)
Mx = max(abs(x))
% Ax is infinite
xave = sum(x(1:10)/10)
title(['HW1, Prob. 2, part a)  x[n]=sin(\pi n)/ sin(\pi n/10)  B. Breinholt'])


% Part b
N = 1000
n = -N:N
x = (1.2).^(-n).*sin(pi*n/4).*(n>=0);
subplot(3,1,3)
stem(-20:20,x(n>=-20 & n<= 20))
title('HW1, Prob. 2, part b)  x[n] = (1.2)^{-n}sin(\pi n /4)u[n]')

Ex = sum(x.^2)
Px = 1/(2*N+1)*Ex
xrmx = sqrt(Px)
Mx = max(abs(x))
Ax = sum(x)
xave = 1/(2*N+1)*Ax

orient tall
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3. a) $s(t) = -\sin(2\pi t)\,\mathrm{rect}(t-2.5)$

b) $s[n] = \frac{1}{\sqrt{3}}\sin\left(\frac{\pi}{3}(n+1)\right)$

Note: This answer assumes the delta functions on the graph have a height of ½.

4. a) i) Not linear. Counter-example:

Consider time $n=5$.

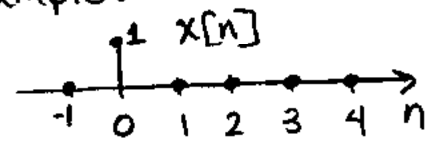
Let $x_1[5]=4$, $x_2[5]=3$, $x_3[n] = x_1[n] + x_2[n] = 7$

Then $y_1[5]=4$, $y_2[5]=3$, $y_3[5]=5$.

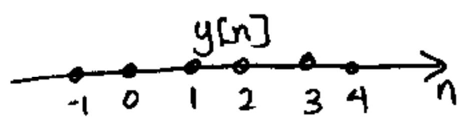
We see $y_3[5] \neq y_1[5] + y_2[5]$. So not linear.

ii) Not TI. Counter example:

Let $x[n] = \delta[n]$

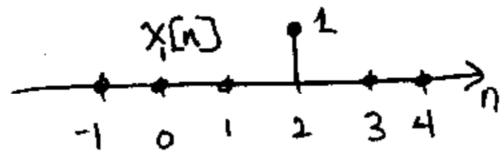


Then $y[n] = 0$



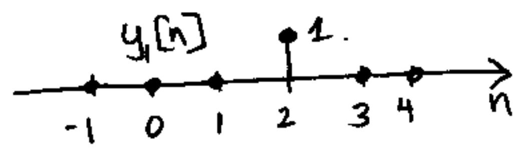
Now, let

$x_1[n] = x[n-2] = \delta[n-2]$



we get

$y_1[n] = \delta[n-2]$



But $y_1[n] \neq y[n-2]$.

iii) Can we get a non-zero output before putting in a non-zero input?

If $x[n]=0$ then $y[n]=0$.

Furthermore, $y[n]$ depends only on current values of n. So answer is no. $\therefore$ Causal

iv) For a bounded input does the system give a bounded output?

Assume $|x[n]| < M_x$ for all n.

There are two cases:

Case 1: If $|n| > M_x$ then $y[n] = x[n] < M_x$.
So the output is bounded.

Case 2: If $|n| < M_x$ then

$$y[n] = \begin{cases} x[n], & x[n] < |n| \\ |n|, & \text{else} \end{cases}$$

$$< \begin{cases} M_x & x[n] < |n| \\ M_x & \text{else} \end{cases}$$

So the output is bounded.

v) The system depends only on current values of $x[n]$.
∴ Memoryless.

vi) Response to an impulse:
If $x[n] = \delta[n]$, $y[n] = 0$.

b) i) Counter-example:

$$x_1[n] = \delta[n] \Rightarrow y_1[n] = \frac{1}{3}\{\delta[n] - \delta[n-1] + 1\}$$

$$x_2[n] = \delta[n-1] \Rightarrow y_2[n] = \frac{1}{3}\{\delta[n-1] - \delta[n-2] + 1\}$$

$$x_3[n] = x_1[n] + x_2[n] \Rightarrow y_3[n] = \frac{1}{3}\{\delta[n] + \delta[n-1] - \delta[n-1] - \delta[n-2] + 1\}$$

$$= \frac{1}{3}\{\delta[n] - \delta[n-2] + 1\}$$

But $y_1[n] + y_2[n] = \frac{1}{3}\{\delta[n] - \delta[n-2] + 2\} \neq y_3[n]$

So not linear.

ii) Time Invariant Proof:

If we put $x_1[n]$ into the system we get

$$y_1[n] = \frac{1}{3}\{x_1[n] - x_1[n-1] + 1\}.$$

If we put $x_2[n] = x_1[n-n_0]$ into the system we get

$$y_2[n] = \frac{1}{3}\{x_2[n] - x_2[n-1] + 1\}$$

$$= \frac{1}{3}\{x_1[n-n_0] - x_1[n-n_0-1] + 1\}$$

We check to see that $y_2[n]$ is a delayed version of $y_1[n]$

$$y_1[n-n_0] = \frac{1}{3}\{x_1[n-n_0] - x_1[n-n_0-1] + 1\}$$

$$= y_2[n].$$

It is, so the system is time invariant.

iii) Causal. The system does not depend on future values of $x[n]$.

iv) Assume $|x[n]| \leq M_x$ for all n.

$$y[n] = \frac{1}{3}\{x[n] - x[n-1] + 1\}$$

$$\leq \frac{1}{3}\{|x[n]| + |x[n-1]| + 1\}$$ by the Triangle inequality.

$$\leq \frac{1}{3}\{M_x + M_x + 1\} = \frac{2}{3}M_x + \frac{1}{3}.$$ Bounded.

So Stable.

v) Not memoryless. Counter example:

If $x[n] = 5$ and $n = 0$

$y[0]$ depends on a previous value of $x[n]$, $x[-1]$.

vi) Response to an impulse

$$y[n] = \frac{1}{3}\{\delta[n] - \delta[n-1] + 1\}.$$

c) i) linear

$$y_1(t) = \int_0^t x_1(\tau)\,d\tau$$

$$y_2(t) = \int_0^t x_2(\tau)\,d\tau$$

Let $x_3(t) = a\,x_1(t) + b\,x_2(t)$ where a and b are constants. Then

$$y_3(t) = \int_0^t (a x_1(\tau) + b x_2(\tau))\,d\tau$$

$$= a\int_0^t x_1(\tau)\,d\tau + b\int_0^t x_2(\tau)\,d\tau$$

$$= a\,y_1(t) + b\,y_2(t)$$

$\therefore$ Linear.

ii) Not time-invariant. Counter example:

If $x_1(t) = \delta(t)$ then $y_1(t) = \int_0^t \delta(\tau)\,d\tau = u(t)$

If $x_2(t) = x_1(t+1) = \delta(t+1)$

then $y_2(t) = \int_0^t \delta(\tau+1)\,d\tau = 0$

$y_2(t) \neq y_1(t+1)$

iii) Causal.

$y(t)$ depends only on values of $x(\tau)$ up to time t.

iv) Not BIBO stable. Counter example:

Let $x(t) = u(t)$

Then $y(t) = \int_0^t u(\tau)\,d\tau$

$= t\,u(t)$

$x(t) \leq 1$ for all time.

But $y(t) = t\,u(t) \to \infty$ as $t \to \infty$.

v) Not memoryless.

$y(5)$ depends on an infinite number of previous values of x, including $x(1.5)$.

vi) Response to an impulse

$$y(t) = \int_0^t \delta(\tau)\,d\tau = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases} = u(t)$$

5 a) i) $h[n] = y[n]$ when $x[n] = \delta[n]$.

$$h[n] = (\delta(n) - \delta(n-1))/2$$

ii) $y[n] = \frac{1}{2}\{x[n] - x[n-1]\}$

$$Y(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X(e^{j\omega})e^{-j\omega}\}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{2}(1 - e^{-j\omega})$$

$$= e^{-j\frac{\omega}{2}}(e^{j\omega/2} - e^{-j\omega/2})\frac{1}{2j}j$$

$$= je^{-j\frac{\omega}{2}}\sin(\omega/2)$$

iii) $|ABC| = |A||B||C|$

$$|H(e^{j\omega})| = |j||e^{-j\frac{\omega}{2}}||\sin(\omega/2)|$$

$$= 1 \cdot 1 \cdot |\sin(\omega/2)| = |\sin(\omega/2)|$$

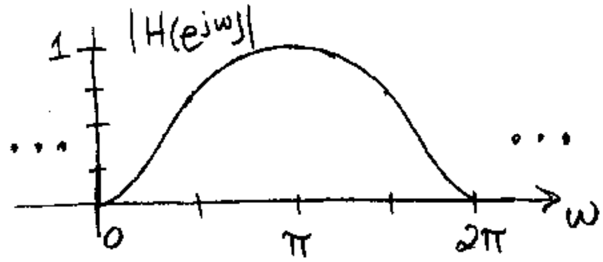
$$\angle ABC = \angle A + \angle B + \angle C$$

$$\angle H(e^{j\omega}) = \angle j + \angle e^{-j\omega/2} + \angle \sin(\omega/2)$$

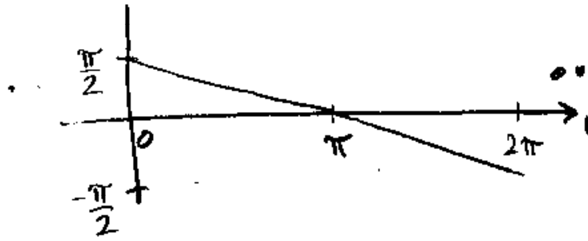
$$= \frac{\pi}{2} - \frac{\omega}{2} + \angle \sin(\omega/2)$$

$$\angle \sin(\omega/2) = \begin{cases} 0 & \text{when } \sin(\omega/2) > 0 \\ \pm\pi & \text{when } \sin(\omega/2) < 0 \end{cases}$$

So $\angle H(e^{j\omega}) = \frac{\pi}{2} - \frac{\omega}{2} + 0$ for $0 \leq \omega \leq 2\pi$



Repeats with period 2π.



Repeats with period 2π.

iv) Attenuates low frequencies (those close to $\omega=0$ or $\omega=2\pi$) while passing high frequencies (those close to $\omega=\pi$). This is a crude high-pass filter.

b) i) $y[n] = (x[n] + y[n-2])/2$.

n	x[n]	$y[n]=\frac{1}{2}(x[n]+y[n-2])$
0	1	$y[0]=\frac{1}{2}(1+y[-2])=\frac{1}{2}$
1	0	$y[1]=\frac{1}{2}(0+y[-1])=0$
2	0	$y[2]=\frac{1}{2}(0+y[0])=\left(\frac{1}{2}\right)^2$
3	0	$y[3]=0$
4	0	$y[4]=\frac{1}{2}(0+y[2])=\left(\frac{1}{2}\right)^3$
5	0	$y[5]=0$
6	0	$y[6]=\left(\frac{1}{2}\right)^4$

To get the impulse response, put in an impulse and see what you get out. Assume the system is causal, i.e. $y[n]=0$ for $n<0$.

An expression that describes this is

$$h[n] = \left(\frac{1}{2}\right)^{\left(\frac{n}{2}+1\right)} \underbrace{\left(1+(-1)^n\right)\frac{1}{2}}_{=\begin{cases}1 & \text{for } n \text{ even}\\ 0 & \text{for } n \text{ odd}\end{cases}} u(n)$$

or $h[n] = \left(\frac{1}{2}\right)^{\left(\frac{n}{2}+2\right)}\left(1+(-1)^n\right) u(n)$

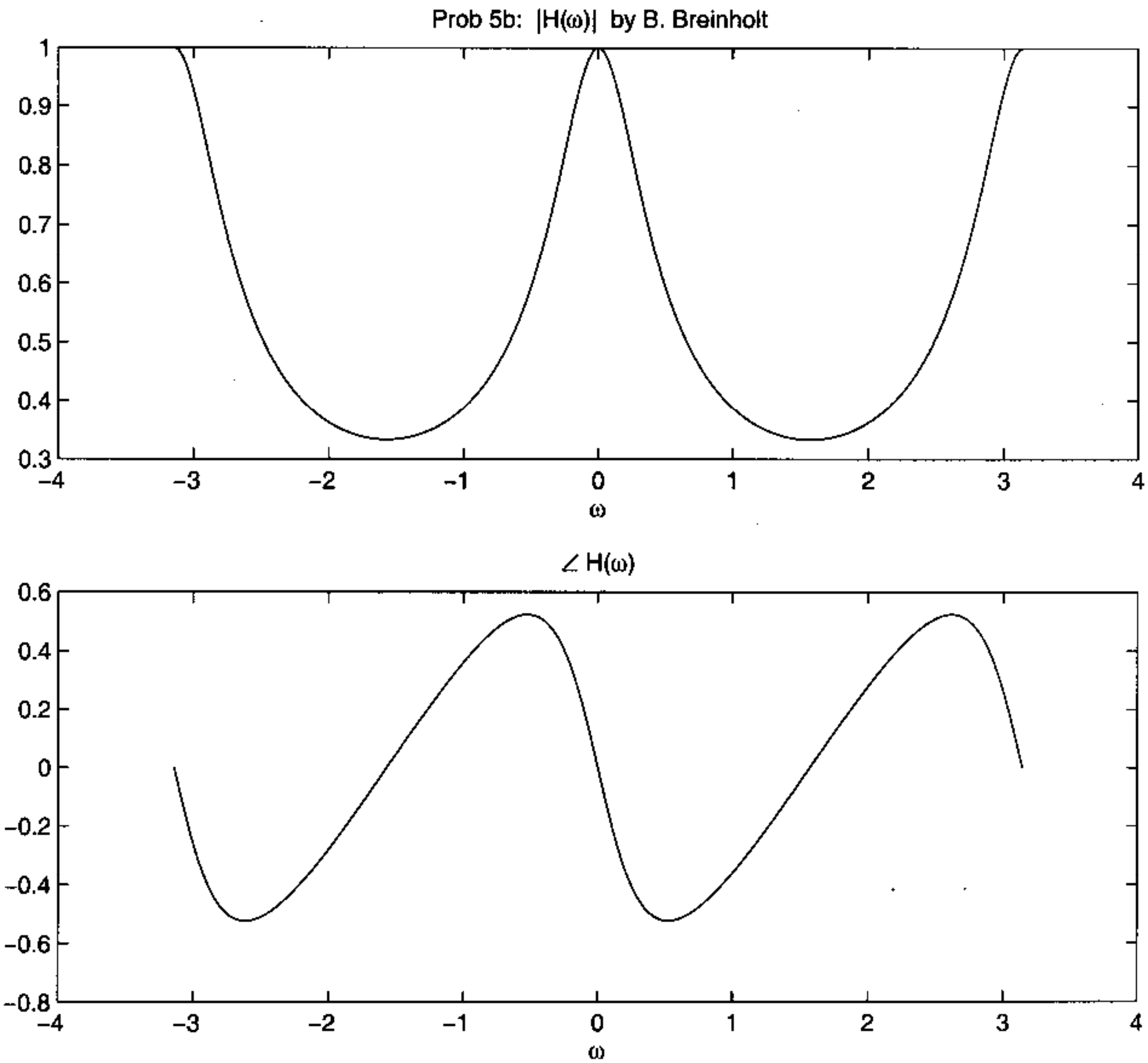
ii) $y[n] = \frac{1}{2}\left(x[n] + y[n-2]\right)$

$$Y(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) + Y(e^{j\omega})e^{-j2\omega}\right)$$

$$Y(e^{j\omega})\left(1 - \frac{1}{2}e^{-j2\omega}\right) = \frac{1}{2}X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\frac{1}{2}}{1-\frac{1}{2}e^{-j2\omega}} = \frac{1}{2-e^{-j2\omega}}$$

iii) Done using MATLAB:



iv) Passes low and high frequencies but attenuates middle frequencies.

Remember that $\omega = \pi$ corresponds to $F = \frac{F_s}{2}$ Hz.

c) i) $h[n] = \{\delta[n] + 2\delta[n-1] + \delta[n-2]\}/4$

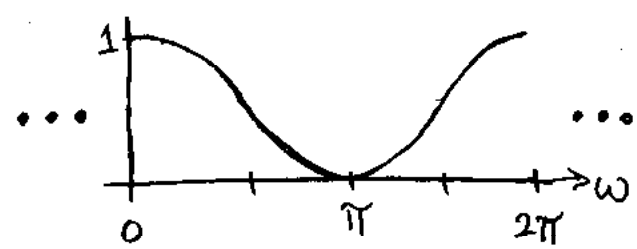
ii) $H(e^{j\omega}) = \frac{1}{4}\{1 + 2e^{-j\omega} + e^{-j2\omega}\}$

$$= \frac{1}{4} e^{-j\omega}\{e^{j\omega} + 2 + e^{-j\omega}\}$$

$$= \frac{1}{4} e^{-j\omega}\left\{2 + 2\frac{e^{j\omega} + e^{-j\omega}}{2}\right\}$$

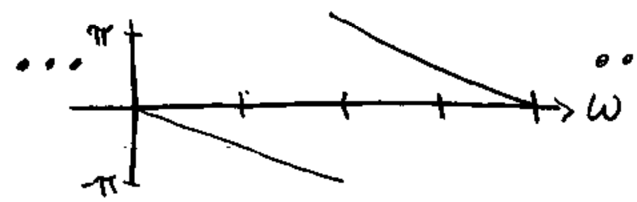
$$= \frac{1}{4} e^{-j\omega}\{2 + 2\cos\omega\} = \frac{1}{2} e^{-j\omega}\{1 + \cos\omega\}$$

iii) $|H(e^{j\omega})| = |\tfrac{1}{2}|\,|e^{-j\omega}|\,|1+\cos\omega| = \tfrac{1}{2}|1+\cos\omega|$



$$\angle H(e^{j\omega}) = \angle\tfrac{1}{2} + \angle e^{-j\omega} + \underbrace{\angle(1+\cos\omega)}_{\text{Always positive \& real. So } \angle = 0}$$

$$= 0 - \omega + 0 = -\omega$$



iv) Passes low frequencies. Attenuates high frequencies. This is a crude low-pass filter.

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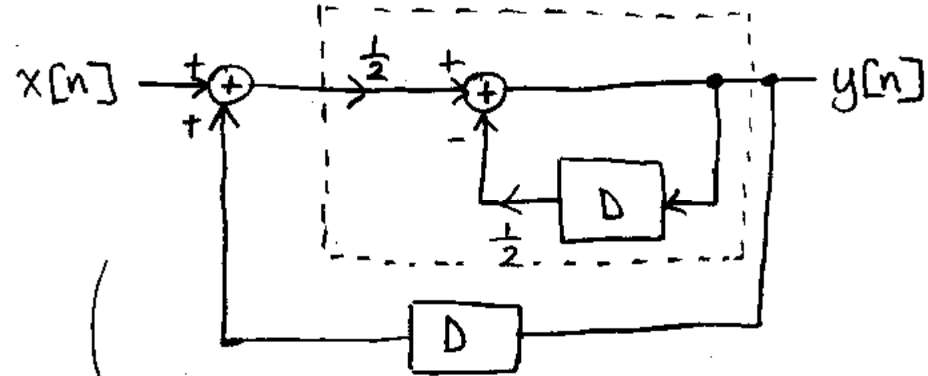
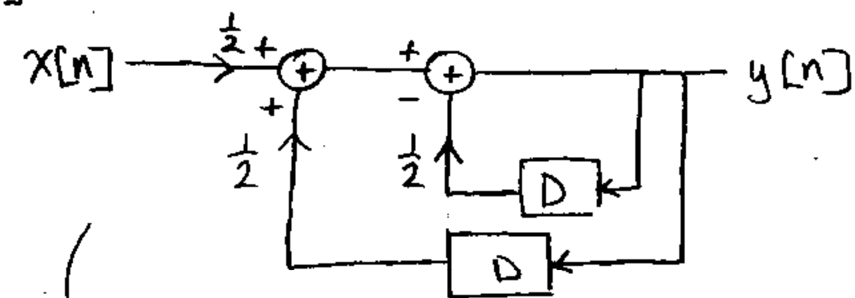
a)

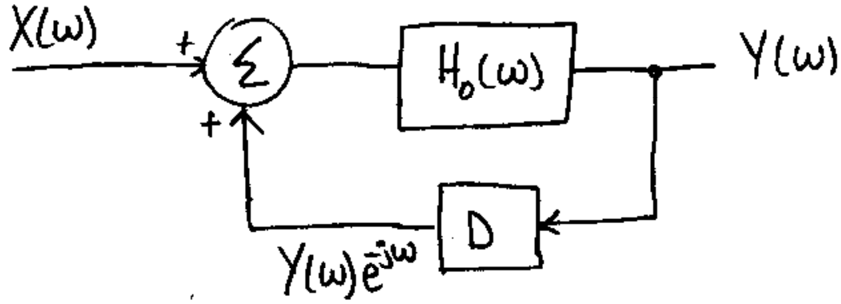




$$x[n] \xrightarrow{\frac{1}{2}} y[n]$$

$$y[n] = \frac{1}{2} x[n]$$

b)



$$Y(\omega) = \left\{ X(\omega) + Y(\omega) e^{-j\omega} \right\} H_0(\omega)$$

$$Y(\omega)\left(1 - H_0(\omega) e^{-j\omega}\right) = X(\omega) H_0(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{H_o(\omega)}{1 - H_o(\omega)e^{-j\omega}} = \frac{1}{\frac{1}{H_o(\omega)} - e^{-j\omega}}$$

c) $H(\omega)$ from part a)

$$Y(\omega) = \frac{1}{2}X(\omega) \Rightarrow H(\omega) = \frac{1}{2}$$

$H(\omega)$ from part b)

1^{st} find $H_o(\omega)$:

$$y_o[n] = \frac{1}{2}\left(x_o[n] - y_o[n-1]\right)$$

$$Y_o(\omega) = \frac{1}{2}X_o(\omega) - \frac{1}{2}Y_o(\omega)e^{-j\omega}$$

$$Y_o(\omega)\left(1 + \frac{1}{2}e^{-j\omega}\right) = \frac{1}{2}X_o(\omega)$$

$$\Rightarrow H_o(\omega) = \frac{1}{2}\cdot\frac{1}{1+\frac{1}{2}e^{-j\omega}} = \frac{1}{2+e^{-j\omega}}$$

Now plug into answer from part b)

$$H(\omega) = \frac{1}{\frac{1}{\left(\frac{1}{2+e^{-j\omega}}\right)} - e^{-j\omega}} = \frac{1}{2+e^{-j\omega} - e^{-j\omega}}$$

$$= \frac{1}{2}$$

So both approaches give the same result.