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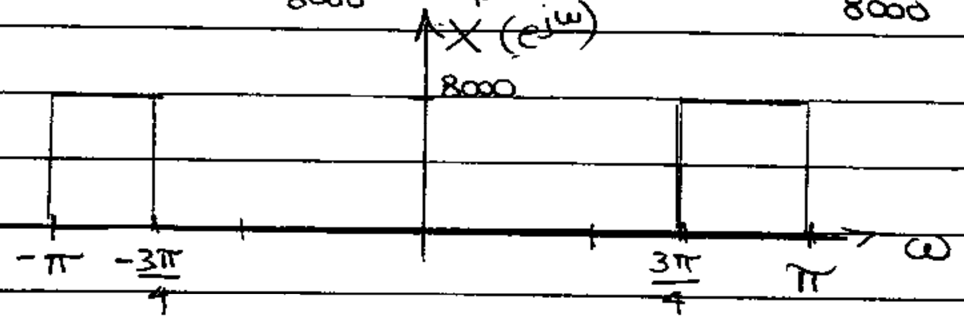
EE438 DSP w/ Applications　　Sp. '01

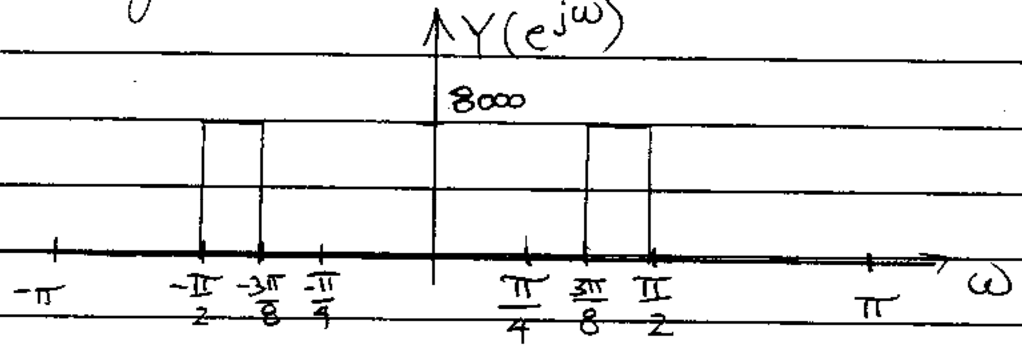
Solutions to Exam 1

Problem 1

(a) $\omega_1 = \frac{2\pi\, 3000}{8000} = \frac{3}{4}\pi$　　$\omega_2 = \frac{2\pi\, 4000}{8000} = \pi$



(b) $y(n) = \frac{1}{2} x_a\left(n/16{,}000\right)$



(c) (i) $a_1x_1(n) + a_2x_2(n) \longrightarrow \underset{\overset{\uparrow}{n}}{\otimes} \longrightarrow a_1 n x_1(n) + a_2 n x_2(n)$

$= a_1 y_1(n) + a_2 y_2(n)$

$\Rightarrow$ Linear!

(ii)

$x(n) \longrightarrow \underset{\overset{\uparrow}{n}}{\otimes} \longrightarrow y(n)$　　$x(n-n_0) \longrightarrow \underset{\overset{\uparrow}{n}}{\otimes} \longrightarrow n\,x(n-n_0)$

$\neq y(n-n_0)$

$= (n-n_0)\,x(n-n_0)$

$\Rightarrow$ not TI

(iii) Unstable $\Rightarrow$ consider $x(n) = u(n)$

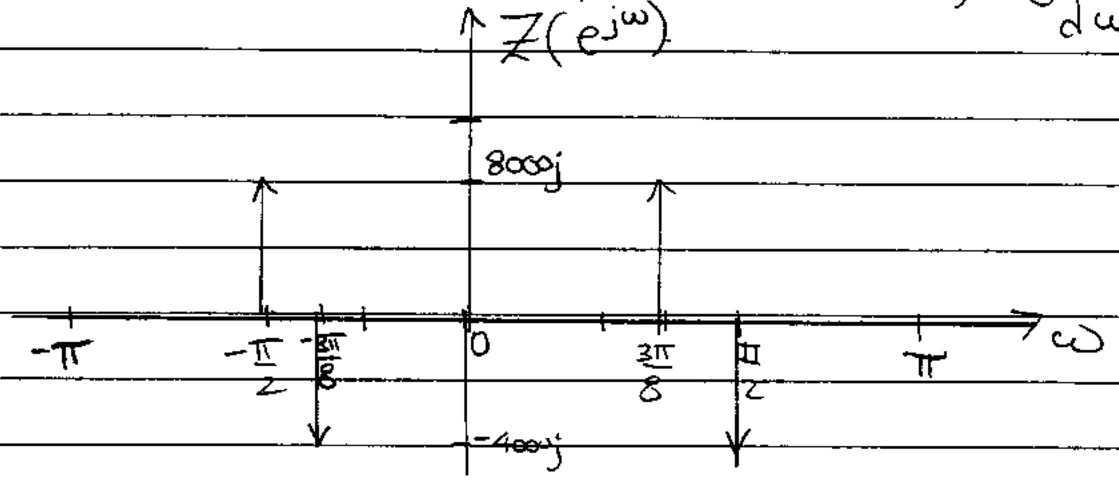
$\Rightarrow y(n) = n\,u(n)$

as $n \to \infty \Rightarrow y(n) \to \infty$

Exam 1 Solutions

Prob. 1 Sol'n. (cont.)

(d) from Table of DTFT props, $Z(e^{j\omega}) = j\frac{d}{d\omega}X(e^{j\omega})$



(e) $x_a(t) = 2000 \text{ sinc}(1000t)\cos(2\pi 3500 t)$

$$x(n) = x_a\left(\frac{n}{8000}\right) = 2000 \text{ sinc}\left(\frac{n}{8}\right)\cos\left(2\pi\frac{35}{80}n\right)$$

$$= 2000 \text{ sinc}\left(\frac{n}{8}\right)\cos\left(\frac{7}{8}\pi n\right)$$

$$y(n) = \frac{1}{2}x_a\left(\frac{n}{16000}\right)$$

$$= 1000 \text{ sinc}\left(\frac{n}{16}\right)\cos\left(\frac{7}{16}\pi n\right)$$

$$z(n) = n\,y(n) = 1000\,\cancel{n}\,\frac{\sin\left(\frac{\pi}{16}n\right)}{\pi\frac{\cancel{n}}{16}}\cos\left(\frac{7}{16}\pi n\right)$$

$$= \frac{16{,}000}{\pi}\sin\left(\frac{\pi}{16}n\right)\cos\left(\frac{7\pi}{16}n\right)$$

$$= \frac{8{,}000}{\pi}\sin\left(\frac{\pi}{2}n\right) - 8{,}000\sin\left(\frac{3\pi}{8}n\right)$$

- could have obtained the same result by taking inverse DTFT of $Z(e^{j\omega})$ above and using DTFT pair: $\sin(\omega_0 n) \xleftrightarrow{DTFT} -j\pi\delta(\omega-\omega_0) + j\pi\delta(\omega+\omega_0)$

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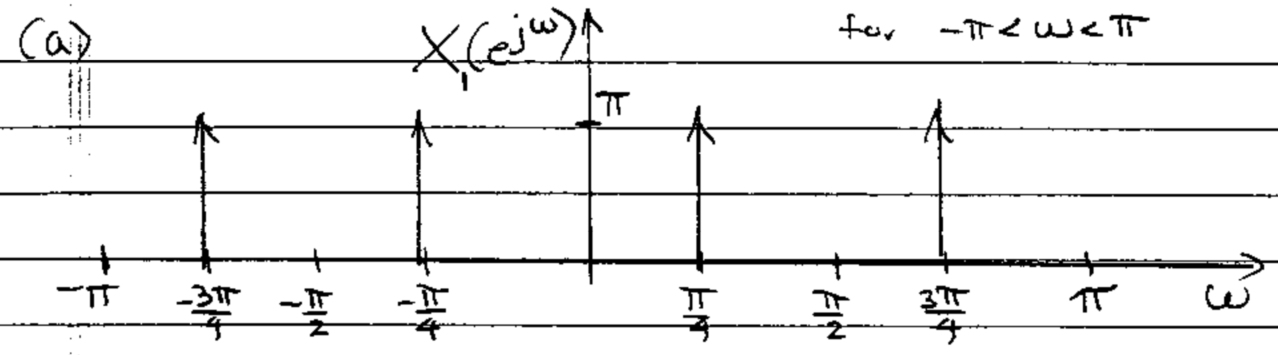
Exam 1 Solutions

Problem 2 : $\cos \omega_0 n \xleftrightarrow{DTFT} \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$

for $-\pi<\omega<\pi$

(a)



$\omega_1 = 2\pi\frac{1000}{8000} = \frac{\pi}{4}$ $\omega_2 = 2\pi\frac{3000}{8000} = \frac{3\pi}{4}$

(b) (i) Not linear $a\,x(n) \longrightarrow \boxed{(\cdot)^2} \longrightarrow a^2x^2(n)$

$\neq a\,y(n)$

$= a\,x^2(n)$

(ii) Time-invariant : $x(n-n_0) \rightarrow \boxed{(\cdot)^2} \rightarrow x^2(n-n_0)$

$= y(n-n_0)$

(iii) Stable ⇒ if $x(n)<\infty$, then $x^2(n)<\infty$

(d) $x_1(n) = \cos\left(\frac{\pi}{4}n\right)+\cos\left(\frac{3\pi}{4}n\right)$

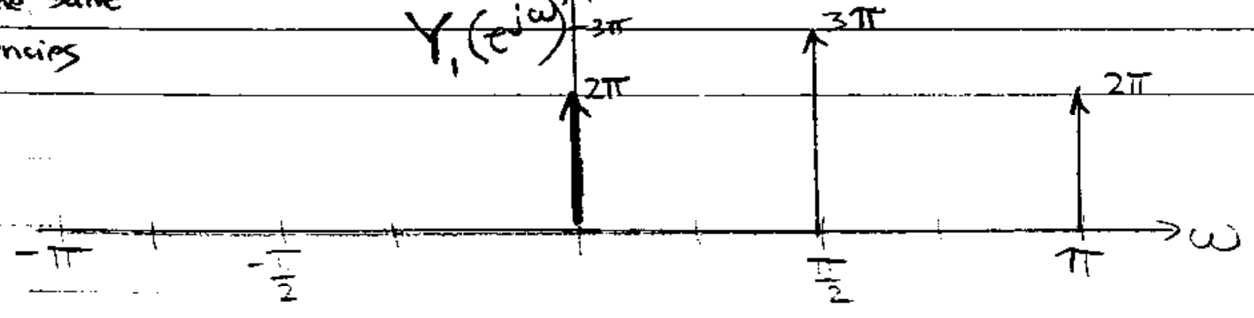
$y_1(n) = x_1^2(n) = \cos^2\left(\frac{\pi}{4}n\right) + 2\cos\left(\frac{\pi}{4}n\right)\cos\left(\frac{3\pi}{4}n\right)+\cos^2\left(\frac{3\pi}{4}n\right)$

$= \frac{1}{2}+\cos\left(\frac{\pi}{2}n\right)+\cos(\pi n)+\cos\left(\frac{\pi}{2}n\right)+\frac{1}{2}+\cos\left(\frac{6\pi}{4}n\right)$

$= 1+2\cos\left(\frac{\pi}{2}n\right)+e^{j\pi n}+\cos\left(\frac{3\pi}{2}n-\frac{4\pi}{2}n\right)$

$= 1+3\cos\left(\frac{\pi}{2}n\right)+e^{j\pi n}$

Remember:
π and −π
are the same
frequencies



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Exam 1 Solutions

Problem 2 (cont.)

$$y_a(t) = x_a^2(t) = 1 + \cos(2\pi 2000t) + \cos(2\pi 2000t) + \cos(2\pi 4000t) + \cos(2\pi 6000t)$$

$$= 1 + 2\cos(2\pi 2000t) + \cos(2\pi 4000t) + \cos(2\pi 6000t)$$

$$Y_a(f) = \delta(f) + \delta(f-2000) + \delta(f+2000) + \tfrac{1}{2}\delta(f-4000) + \tfrac{1}{2}\delta(f+4000) + \tfrac{1}{2}\delta(f-6000) + \tfrac{1}{2}\delta(f+6000)$$

$$X_2(e^{j\omega}) = f_s X_a\left(\frac{f_s \omega}{2\pi}\right) \qquad f_s = 8\text{ KHz}$$

$$= 8K\,\delta\left(\frac{8K\omega}{2\pi}\right) + 8K\,\delta\left(\frac{8K\omega}{2\pi} - 2K\right) + 8K\,\delta\left(\frac{8K\omega}{2\pi} + 2K\right) + 4K\,\delta\left(\frac{8K\omega}{2\pi} - 4K\right) + 4K\,\delta\left(\frac{8K\omega}{2\pi} + 4K\right) + 4K\,\delta\left(\frac{8K\omega}{2\pi} - 6K\right) + 4K\,\delta\left(\frac{8K\omega}{2\pi} + 6K\right)$$

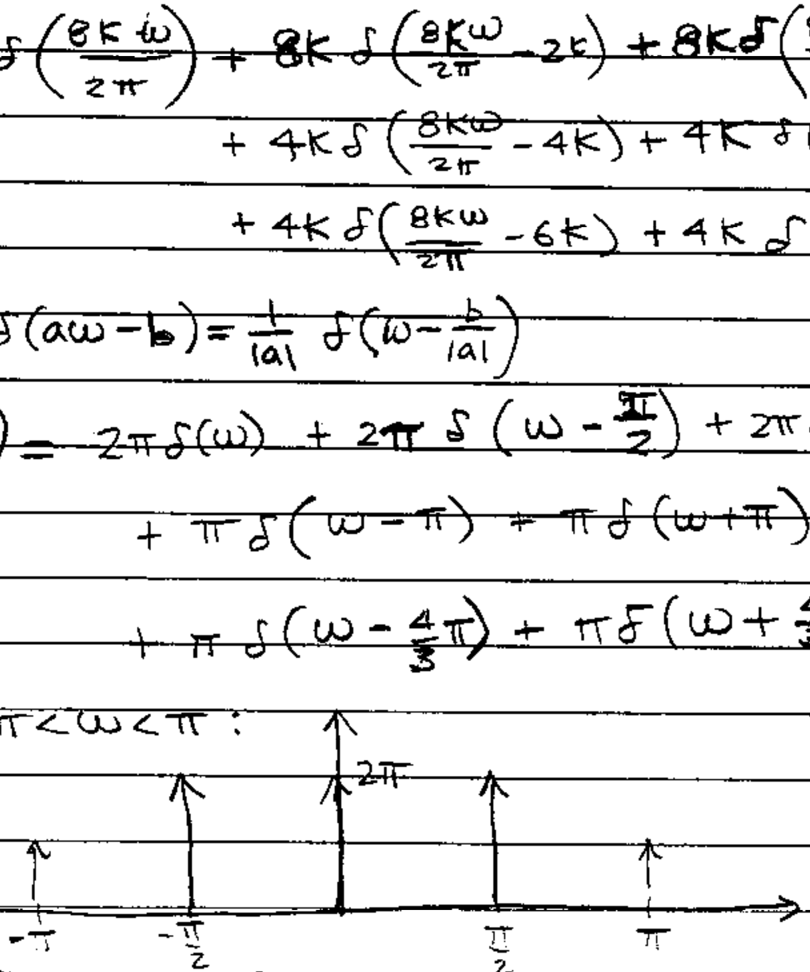
using $\delta(a\omega - b) = \frac{1}{|a|}\delta\left(\omega - \frac{b}{|a|}\right)$

$$X_2(e^{j\omega}) = 2\pi\delta(\omega) + 2\pi\,\delta\left(\omega - \frac{\pi}{2}\right) + 2\pi\,\delta\left(\omega + \frac{\pi}{2}\right)$$

$$+ \pi\,\delta(\omega - \pi) + \pi\,\delta(\omega + \pi) \quad \text{\} same freq.}$$

$$+ \pi\,\delta\left(\omega - \frac{4}{3}\pi\right) + \pi\,\delta\left(\omega + \frac{4}{3}\pi\right)$$

over $-\pi < \omega < \pi$:



$X_2(e^{j\omega}) \neq Y_1(e^{j\omega})$ because highest freq. in $y_a(t) = x_a^2(t)$ is 6 KHz and fs = 8 KHz is below required Nyquist rate of 12KHz

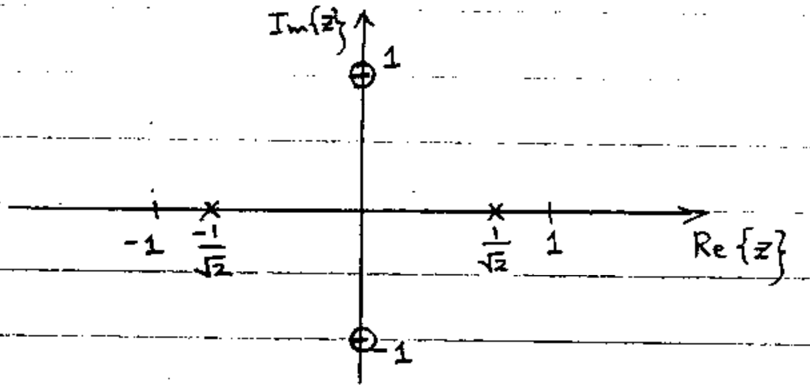
Solution to Problem 3

(a) $Y(z)\left\{1-\frac{1}{2}z^{-2}\right\} = X(z)\left\{1+z^{-2}\right\}$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2+1}{z^2-\frac{1}{2}} = \frac{(z+j)(z-j)}{\left(z+\frac{1}{\sqrt{2}}\right)\left(z-\frac{1}{\sqrt{2}}\right)}$$

(b) zeroes: $z_1 = -j = e^{-j\frac{\pi}{2}}$; $z_2 = j = e^{j\frac{\pi}{2}}$

poles: $z_1 = \frac{1}{\sqrt{2}}$; $z_2 = -\frac{1}{\sqrt{2}}$



ROC:
$|z| > \frac{1}{\sqrt{2}}$

- ROC: $|z| > \frac{1}{\sqrt{2}}$ includes unit circle, $|z| = 1$, so system is stable

(c) $H(e^{j\omega})\Big|_{\omega=0} = H(z)\Big|_{z=1} = \frac{z^2+1}{z^2-\frac{1}{2}}\Big|_{z=1} = 4$

$\Rightarrow |H(e^{j0})| = 4$

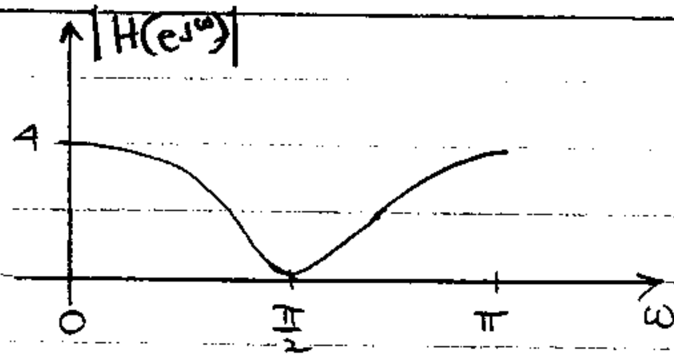
$H(e^{j\omega})\Big|_{\omega=\frac{\pi}{2}} = H(z)\Big|_{z=j} = \frac{z^2+1}{z^2-1/2}\Big|_{z=j} = 0$ $\Big\}$ zero on unit circle $z = e^{j\pi/2}$

$\Rightarrow |H(e^{j\pi/2})| = 0$

$H(e^{j\omega})\Big|_{\omega=\pi} = H(z)\Big|_{z=-1} = \frac{z^2+1}{z^2-\frac{1}{2}}\Big|_{z=-1} = \frac{2}{\frac{1}{2}} = 4$

$\Rightarrow |H(e^{j\pi})| = 4$

Solution to Prob. 3 (conti.)



(d) $$\frac{H(z)}{z} = \frac{z^2+1}{z\left(z-\frac{1}{\sqrt{2}}\right)\left(z+\frac{1}{\sqrt{2}}\right)} = \frac{A}{z} + \frac{B}{z-\frac{1}{\sqrt{2}}} + \frac{C}{z+\frac{1}{\sqrt{2}}}$$

$$A = \left.\frac{z^2+1}{z^2-\frac{1}{2}}\right|_{z=0} = \frac{1}{-\frac{1}{2}} = -2$$

$$B = \left.\frac{z^2+1}{z\left(z+\frac{1}{\sqrt{2}}\right)}\right|_{z=\frac{1}{\sqrt{2}}} = \frac{3/2}{\left(\frac{1}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{2}}\right)} = \frac{3}{2}$$

$$C = \left.\frac{z^2+1}{z\left(z-\frac{1}{\sqrt{2}}\right)}\right|_{z=-\frac{1}{\sqrt{2}}} = \frac{3/2}{\left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-2}{\sqrt{2}}\right)} = \frac{3}{2}$$

$$h(n) = -2\,\delta(n) + \frac{3}{2}\left(\frac{1}{\sqrt{2}}\right)^n u(n) + \frac{3}{2}\left(\frac{-1}{\sqrt{2}}\right)^n u(n)$$
$$= -2\,\delta(n) + \frac{3}{2}\left(\sqrt{2}\right)^{-n}\left\{1+(-1)^n\right\} u(n)$$

(e) $$x(n) = \frac{1}{2} e^{j\frac{\pi}{2}n} + \frac{1}{2} e^{-j\frac{\pi}{2}n} + e^{j\pi n}$$

$$\Rightarrow y(n) = \frac{1}{2} \underbrace{H\left(e^{j\frac{\pi}{2}}\right)}_{\phi} e^{j\frac{\pi}{2}n} + \frac{1}{2} \underbrace{H\left(e^{-j\frac{\pi}{2}}\right)}_{\phi} e^{-j\frac{\pi}{2}n} + H\left(e^{j\pi}\right) e^{j\pi n}$$

$$= (4)\, e^{j\pi n} = 4\,(-1)^n$$

$$\boxed{y(n) = 4\,(-1)^n}$$