BALAKRISHNAN/RUNDELL
SEP 24, 2002

ECE 301 Midterm Examination #1

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!** Also circle your section.
2. This exam has two parts.

   Part I consists of questions for which no justification is required. Enter the answers to Part I in the spaces provided. **Partial credit will not be provided for problems from Part I.**

   Part II consists of three problems. Unless otherwise instructed, justify your answers to these problems completely. Please note that answers provided without justification to those problems requiring a full justification will be given zero credit.
3. This exam is worth 100 points. You have **one hour** to complete it.
4. There are **11** pages in the exam booklet. Use the back of each page for rough work, if necessary.
5. Please work as neatly as you can.
6. No calculators or crib-sheets are allowed.
7. You might want to read through all of the problems first, to get a feel for how long each one might take, but don't worry—several of the questions are easier than they might appear when you just scan them. Good luck!

IMPORTANT! Whenever a certain space is provided for the final answer, be sure to enter your answer there.

Name:		
Student ID #:	SOLUTIONS	
E-mail address:		
Section (circle one):	Section 1 Rundell	Section 2 Balakrishnan
Signature:		

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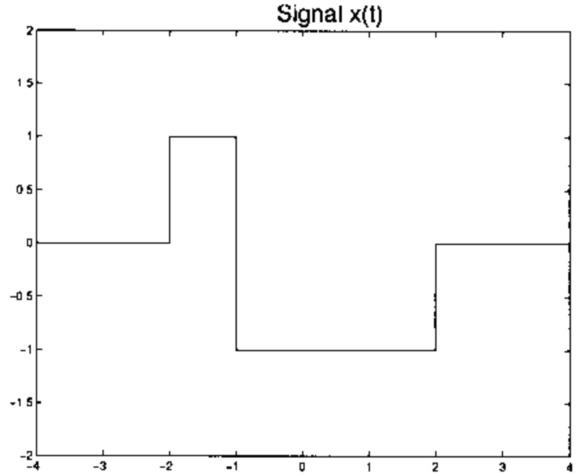
Questions for Part I

Do not justify your answer.

Partial credit is NOT available.

1. (30 points) A continuous-time signal $x(t)$ is shown below. (The signal is zero over the time intervals not shown in the figure.)



(a) (4 points) What is the energy in $x(t)$ over the infinite interval, that is, what is E_∞?

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-2}^{-1} (1)^2 dt + \int_{-1}^{2} (-1)^2 dt$$

$$= 4$$

(b) (2 points) What is the power in $x(t)$ over the infinite interval, that is, what is P_∞?

As $E_\infty < \infty$, $P_\infty = 0$

②

Six signals labeled "Signal A" through "Signal F" are shown on the next page. (The signal $x(t)$ is also shown at the top for your convenience.) Match these signals with the following six signals, and enter your answers in the appropriate spaces in the following table. Each entry must be a letter from "A" through "F". Each answer is worth four points.

(c) The even part of $x(t)$, that is, $x_{\text{even}}(t)$.

(d) The odd part of $x(t)$, that is, $x_{\text{odd}}(t)$.

(e) $x(1-t)$.

(f) $x(-1-t)$.

(g) $x(1-2t)$.

(h) $x(2-2t)$.

Table I

(c) $x_{\text{even}}(t)$	(d) $x_{\text{odd}}(t)$	(e) $x(1-t)$	(f) $x(-1-t)$	(g) $x(1-2t)$	(h) $x(2-2t)$
F	E	C	A	D	B

(Easily verified)

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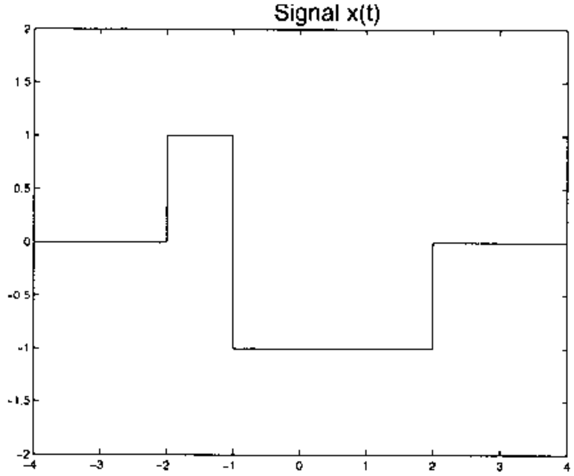
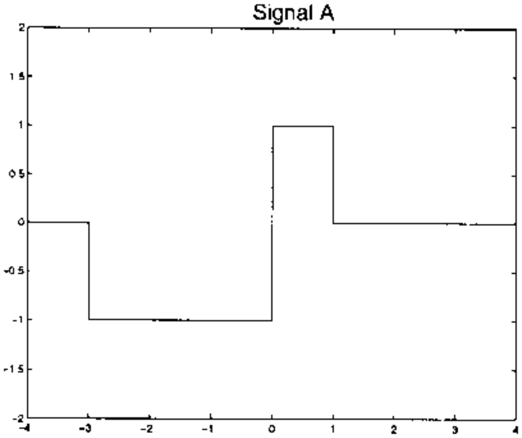
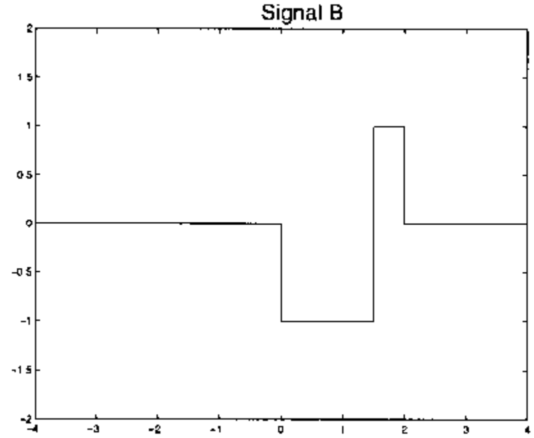
Signal x(t)



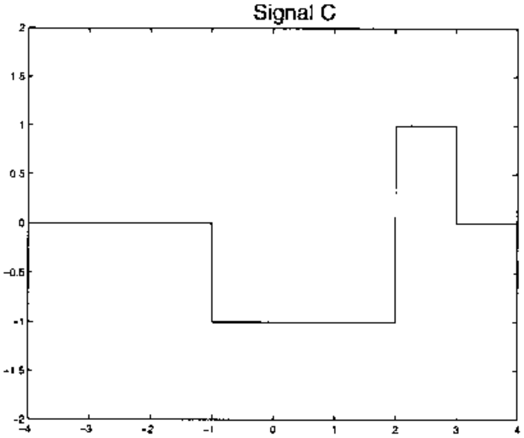
Signal A



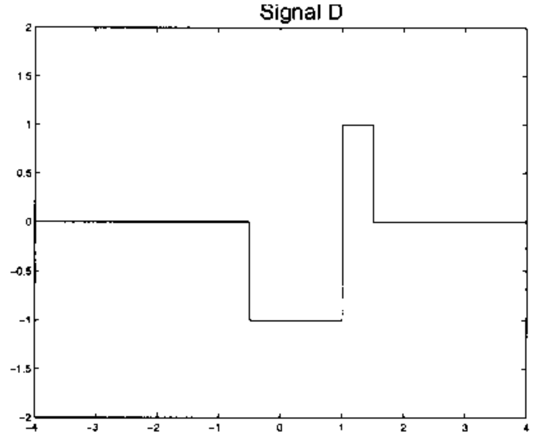
Signal B



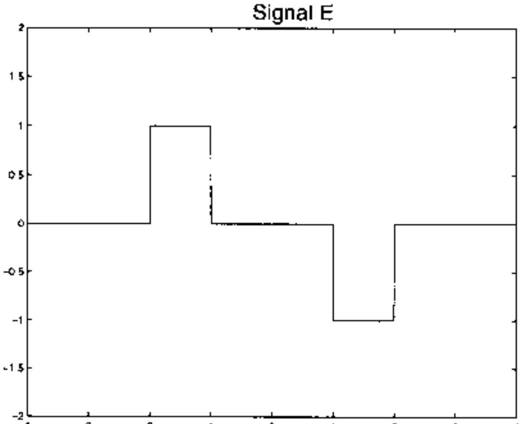
Signal C



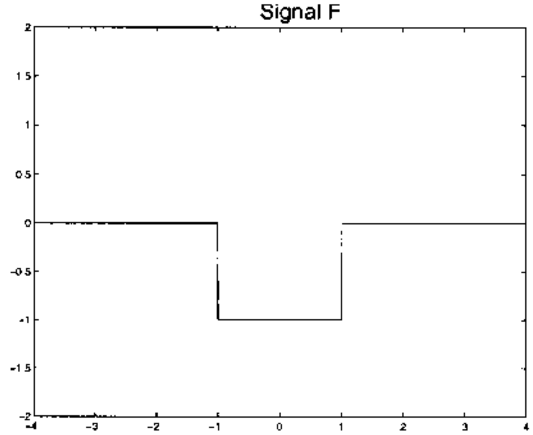
Signal D



Signal E



Signal F



2. (20 points) Determine if each of the following systems (with input x and output y) is memoryless or with-memory; invertible or non-invertible; causal or non-causal; stable or unstable; time-invariant or time-varying; linear or nonlinear.

   (a) **System I**:

$$y(t) = \int_0^t x(t-\tau)e^{-\tau}\,d\tau.$$

   (b) **System II**:

$$y(t) = x(\sin(t)).$$

Enter your answers in the following table. In each entry in the table, write "**Y**" if you can conclude that the property listed on the left holds for the system listed at the top of the column. Enter "**N**" if it can be concluded that the property does not hold. (For each case, there is sufficient data to make this decision.)

	System I	System II
Memoryless	N	N
Invertible	Y	N
Causal	N	N
Stable	N	Y
Time-invariant	N	N
Linear	Y	Y

Every entry in the table is worth 1.5 points. You will get two additional points if all the entries are correct.

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2.

ⓐ <u>System I</u>

$$y(t) = \int_0^t x(t-\tau)e^{-\tau}d\tau$$

<u>Memoryless</u>? $\boxed{\text{No,}}$ because $y(t)$ depends on $\{x(t) : t \in [0,t]\}$

<u>Invertible</u>? $\boxed{\text{Yes.}}$ Write

$$y(t) = \int_0^t x(\tau)\, e^{-(t-\tau)}\, d\tau \quad \text{(change of variables)}$$

$$= e^{-t} \int_0^t x(\tau) e^{\tau} d\tau \quad \text{..... } (*)$$

Then,

$$\frac{dy}{dt} = -y(t) + x(t)$$, or can get $x(t)$

from $y(t)$ using $x(t) = y(t) + \frac{dy}{dt}(t)$

<u>Causal</u>? $\boxed{\text{No.}}$ For example, $y(-1)$ depends on $\{x(t) : t \in [-1, 0]\}$

Stable? $\boxed{\text{No.}}$ Consider bounded input

$$x(t) = u(-t)$$

Then, using equation (*) from previous page,

$$y(t) = e^{-t} \int_0^t x(\tau) e^{\tau} d\tau ,$$

which for $t<0$ becomes

$$y(t) = e^{-t} \int_0^t (1) e^{\tau} d\tau$$

$$= 1 - e^{-t} \xrightarrow{t \to -\infty} -\infty$$

(unbounded)

Time-invariant? $\boxed{\text{No.}}$

Take $x(t) = \delta(t + 1/2)$, then $y(1) = 0$

Shift input to right by 1, and consider

$$\tilde{x}(t) = x(t-1) = \delta(t - 1/2)$$

If system were time-invariant, new $\tilde{y}(2)$ must equal $y(1)$ (the old output)

But $\tilde{y}(2) = e^{-2} \int_0^2 \delta(\tau - 1/2) e^{\tau} d\tau \neq 0$

Linear? $\boxed{\text{Yes.}}$ Easily verified.

(b) System II

$$y(t) = x(\sin(t))$$

(Key: $y(t)$ only depends on $\{x(t) : t \in [-1, 1]\}$)

Memoryless? $\boxed{\text{No}}$ $y(2\pi) = x(\sin(2\pi))$
$= x(0)$

Invertible? $\boxed{\text{No.}}$

$x(t) = 0$ for all $t \longmapsto y(t) = 0$ for all t

$x(t) = u(t-2) \longmapsto y(t) = 0$ for all t

So can't tell these two inputs apart from just looking at $y(t)$.

<u>Causal</u>? $\boxed{\text{No.}}$ $y(-2\pi) = x(0)$
("future" input)

<u>Stable</u>? $\boxed{\text{Yes.}}$ $|y(t)| \le |x(\sin(t))|$

$$\le \max_{t \in [-1,1]} |x(t)|$$

$$\le \max_{t \in \mathbb{R}} |x(t)|$$

<u>Time-invariant</u>? $\boxed{\text{No.}}$

Input $u(t-2) \longmapsto$ Zero output

Shifted input

$u(t+2) \longmapsto$ Nonzero output
(in fact $y(t) = 1$ for all t)

<u>Linear</u>? $\boxed{\text{Yes.}}$ Easily verified

3. (10 points) Circle whether each of the statements is true or false, following these instructions:

- Do not justify your answer.
- A statement is true if it is always true, without further qualifications. It is false otherwise.

(a) (2 points) If $y(t)$ is the output of a linear time-invariant system for an input $x(t)$, then $y(-t)$ is the output for the input $x(-t)$.

True **False**

Easy to find counterexamples.
In fact, this was on the last homework!

(b) (2 points) For an unstable system, every bounded input $x(t)$ yields an output that is not bounded.

True **False**

Take an LTI unstable system, with zero input.
Output is bounded (in fact zero)

(c) (2 points) If $x(t)$ is a periodic signal, then $x(t)+x(at)$ is periodic for any real number a.

True **False**

"a" must be rational.

(d) (2 points) If $x[n]$ is a periodic signal, then $x[n]+x[an]$ is periodic for any integer a.

True False

The ratio of the periods of $x[n]$ and $x[an]$ is rational.

(e) (2 points) Let $y(t)$ be the output of a linear time-invariant system for a nonzero input $x(t)$. It is possible to deduce the impulse response $h(t)$ from this information.

True **False**

Suppose $x(t) = 1$ for all t.

Then $y(t) = A$ (a constant) for all t,
where $A = \int_{-\infty}^{\infty} h(\tau)\,d\tau$. Can't determine $h(t)$ from just this!

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Questions for Part II

Justify your answer completely. No credit will be given for answers without justification.

Partial credit is available.

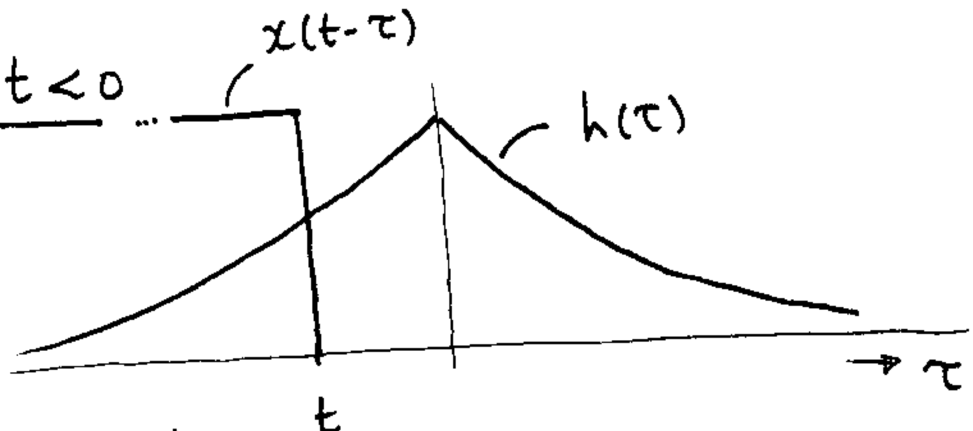
4. (15 points) Consider a linear time-invariant system with impulse response

$$h(t) = e^{-|t|} = \begin{cases} e^{t}, & t < 0, \\ e^{-t}, & t \geq 0. \end{cases}$$

(a) (10 points) Find the step response $s(t)$, i.e., the output of the system when the input is the unit step function $u(t)$.

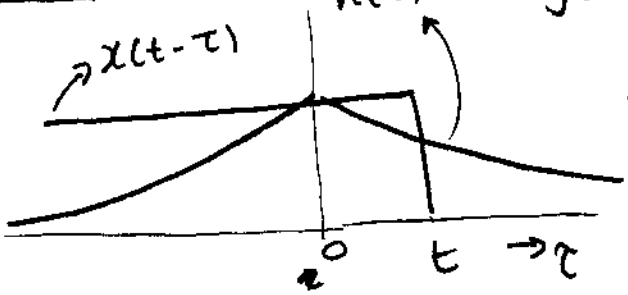
Let us use $y(t) = \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau$

Case $t<0$



$$y(t) = \int_{-\infty}^{t} e^{\tau}\,d\tau = \underline{\underline{e^{t}}}$$

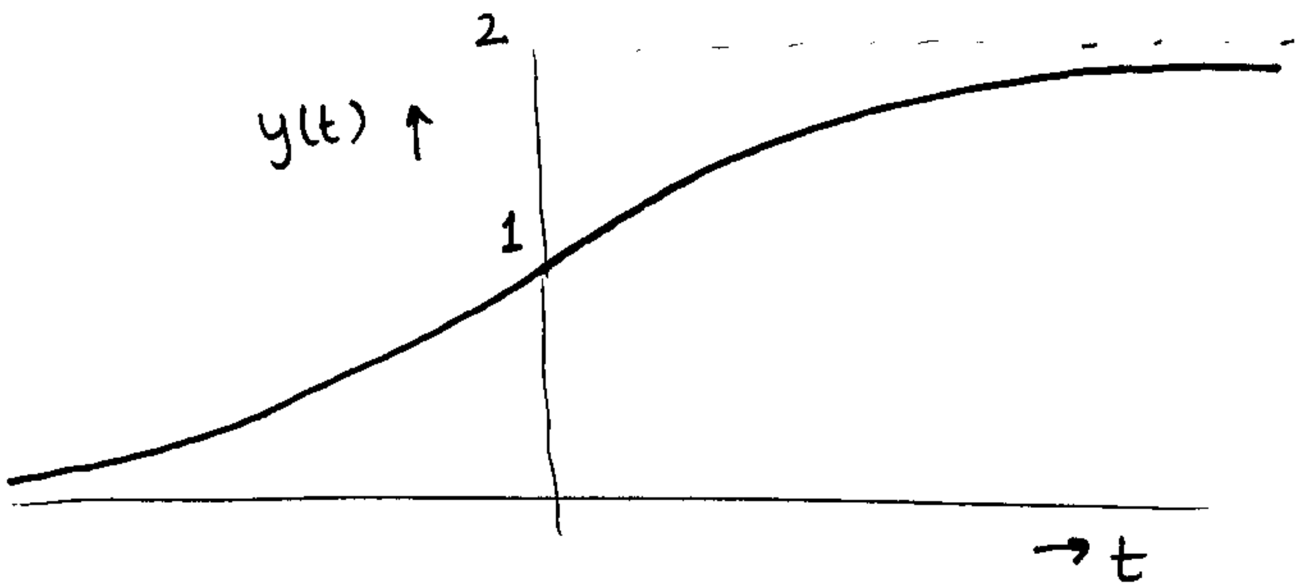
Case $t \geq 0$



$$y(t) = \int_{-\infty}^{t} h(\tau)\,d\tau = \int_{-\infty}^{0} e^{\tau}\,d\tau + \int_{0}^{t} e^{-\tau}\,d\tau = \underline{\underline{2 - e^{-t}}}$$

Plot of y(t)

$$y(t) = \begin{cases} e^{t}, & t < 0 \\ 2 - e^{-t}, & t \geq 0 \end{cases}$$



(b) (2 points) Suppose the input to the system is

$$x(t) = u(t) - u(t-1).$$

Express the output $y(t)$ in terms of the step response $s(t)$.

From linearity & time-invariance,

$$y(t) = s(t) - s(t-1)$$

(c) (3 points) Explicitly write out, as a function of time, the output $y(t)$ that you obtained in part (b) above.

From part (a)

$$s(t) = \begin{cases} e^{t}, & t < 0 \\ 2 - e^{-t}, & t \geq 0 \end{cases}$$

$$s(t-1) = \begin{cases} e^{t-1}, & (t-1) < 0 \\ 2 - e^{-(t-1)}, & (t-1) \geq 0 \end{cases}$$

So $s(t) - s(t-1)$

$$= \begin{cases} e^{t} - e^{(t-1)}, & t < 0 \\ 2 - e^{-t} - e^{+(t-1)} & 0 \leq t < 1 \\ e^{-(t-1)} - e^{-t} & 1 \leq t \end{cases}$$

5. (15 points) The impulse response of a discrete-time linear time-invariant system is given by

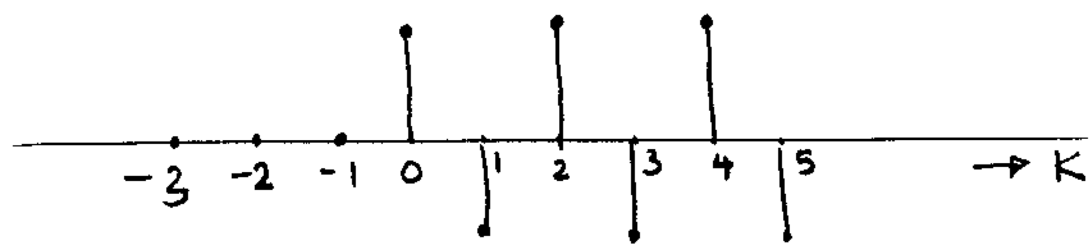
$$
\begin{aligned}
h[n] &= \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3] + \cdots \\
&= \sum_{k=0}^{\infty} (-1)^k \delta[n-k].
\end{aligned}
$$

(a) (5 points) Find the step response $s[n]$ of this system, that is, the output when the input is the unit step function $u[n]$.

Let us use

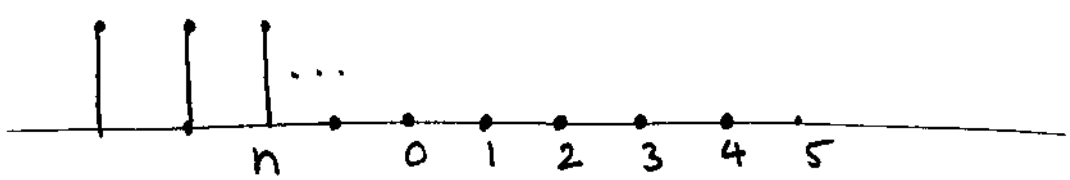
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$h[k]$



Case $n<0$

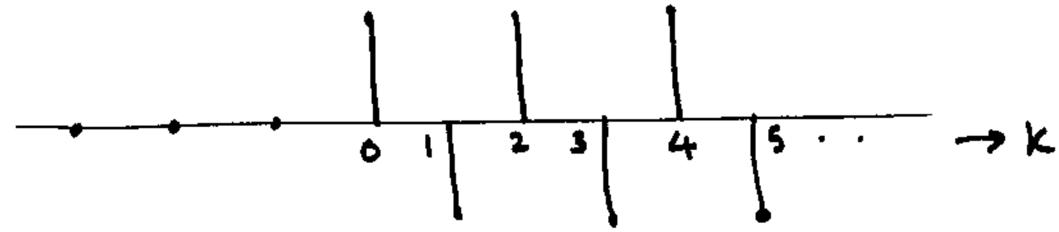
$x[n-k]$



So $y[n] = 0$

Case $n > 0$, and even

$h[k]$



$x[n-k]$

($n=4$ shown here). It should be clear that

$$\sum h[k]\, x[n-k] = 1$$

Case $n > 0$, and odd

Same argument as before, but it should be clear that $\sum h[k]\, x[n-k] = 0$

So

$$y[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \text{ and even} \\ 0, & n \geq 0 \text{ and odd} \end{cases}$$

5. (a) Alternate method

$$u[n] * h[n]$$

$$= u[n] * \left[\delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3] + \cdots \right]$$

$$= \left(u[n] - u[n-1]\right) + \left[u[n-2] - u[n-3]\right] + \cdots$$

$$= \delta[n] + \delta[n-2] + \delta[n-4] + \cdots$$

(same answer as before)

5. (b) $x[n] = \delta[n] + \delta[n-1]$

$$h[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3) + \cdots$$

So $\quad x[n] * h[n]$

$$= \left[\delta[n] + \delta[n-1]\right] * \left[\delta[n] - \delta[n-1) + \delta[n-2] - \cdots\right]$$

$$= \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3] + \cdots$$
$$+ \delta[n-1) - \delta[n-2] + \delta[n-3) - \cdots$$

$$= \delta[n]$$

(can also show this graphically)

(b) (5 points) Find the output $y([n]$ when the input is

$$x[n] = \delta[n] + \delta[n-1].$$

$y[n] = \delta[n]$ (see previous page for work)

(c) (5 points) What is the impulse response of the *inverse* system?

From part (b) $h[n] * [\delta[n] + \delta[n-1]] = \delta[n]$

So $h_{inv}[n] = \delta[n] + \delta[n-1]$

6. (10 points) The Fourier series representation (FSR) of a signal $x(t)$ is

$$x(t) = \sum_{k=0}^{\infty} (0.5)^k e^{jkt}.$$

(a) (2 points) What is the fundamental period of $x(t)$?

$\omega_0 = 1$ (by inspection)

$T = \frac{2\pi}{\omega_0} = 2\pi$

(b) (4 points) Find the Fourier series representation of the even part of $x(t)$, i.e., the FSR of

$$x_{\text{even}}(t) = \frac{1}{2}(x(t) + x(-t)).$$

$x(t) = \sum_{k=0}^{\infty} (0.5)^k e^{jkt}$, so $x(-t) = \sum_{k=0}^{\infty} (0.5)^k e^{-jkt} = \sum_{k=-\infty}^{0} (0.5)^{-k} e^{jkt}$

Thus

$$x_{even}(t) = \frac{\sum_{k=-\infty}^{0} (0.5)^{-k} e^{jkt} + \sum_{k=0}^{\infty} (0.5)^k e^{jkt}}{2}$$

or $a_k^{(even)} = \begin{cases} \frac{1}{2}(0.5)^{-k}, & k < 0 \\ 1, & k = 0 \\ \frac{1}{2}(0.5)^k, & k > 0 \end{cases}$

(c) (4 points) Find the Fourier series representation of the odd part of $x(t)$, i.e., the FSR of

$$x_{\text{odd}}(t) = \frac{1}{2}(x(t) - x(-t)).$$

$$x_{odd}(t) = -\sum_{k=-\infty}^{0} (0.5)^{-k} e^{jkt} + \sum_{k=0}^{\infty} (0.5)^k e^{jkt}$$

So

$$a_k^{(odd)} = \begin{cases} -\frac{1}{2}(0.5)^{-k}, & k < 0 \\ 0, & k = 0 \\ \frac{1}{2}(0.5)^k, & k > 0 \end{cases}$$