Name: ________________

Instructions:

- Follow all instructions carefully!
- This is a 60 minute exam containing 4 problems totaling 120 points.
- You may only use your brain and a pencil (or pen) to complete this exam. You may not use your book, notes or a calculator.

Good Luck.
Name: ____________________________

**Problem 1. (30pt) LTI Systems**

**a)** Consider an LTI system with impulse response \( h(t) = e^{-at}u(t) \) and input \( x(t) = u(t) \). Compute the output \( y(t) \).

\[
\begin{align*}
y(t) &= \int_{-\infty}^{\infty} x(t - \tau) h(\tau) \, d\tau \\
&= \int_{-\infty}^{\infty} u(t - \tau) e^{-a\tau} u(\tau) \, d\tau \\
&= \left( \int_{0}^{t} e^{-a\tau} \, d\tau \right) u(t) \\
&= \frac{-1}{a} e^{-at} \bigg|_{0}^{t} u(t) \\
&= \frac{1}{a} (e^{-at} - 1) u(t) \\
&= \left(1 - e^{-at}\right) u(t)
\end{align*}
\]

**b)** Negate the following statement

\[
\forall x, Px \Rightarrow Qx
\]

\[
\exists x \text{ s.t. } P_x \not\Rightarrow \neg Qx
\]

**b)** Consider the following definition: "A signal \( x(t) \) is **bounded** if there exists an \( M > 0 \) such that for all time \( t \), \( |x(t)| < M \)."

Give the definition of an **unbounded** signal.

**A signal** \( x(t) \) **is unbounded** if for all \( M > 0 \) there exists a \( t \) such that \( |x(t)| \geq M \).
Problem 2. (30pt) LTI System Properties

Consider the following LTI system

\[ y(t) = \int_{0}^{\infty} g(\tau) x(t - \tau) d\tau \]

a) Prove that the system is linear.

b) Compute the system’s impulse response.

c) Is the system, memoryless? Why?

d) Is the system, causal? Why?

\[ a) \quad \text{Let} \quad y_1(t) \equiv \mathcal{L}[x_1(t)] \]
\[ y_2(t) \equiv \mathcal{L}[x_2(t)] \]

Then \( \forall \alpha, \beta, x_1(t) \text{ and } x_2(t) \)

\[ \mathcal{L}\left[ \alpha x_1(t) + \beta x_2(t) \right] \]

\[ = \int_{0}^{\infty} g(\tau) (\alpha x_1(t-\tau) + \beta x_2(t-\tau)) d\tau \]

\[ = \alpha \int_{0}^{\infty} g(\tau) x_1(t-\tau) d\tau + \beta \int_{0}^{\infty} g(\tau) x_2(t-\tau) d\tau \]

\[ = \alpha \mathcal{L}[x_1(t)] + \beta \mathcal{L}[x_2(t)] \]

Q.E.D.

\[ b) \quad h(t) = \mathcal{L}[\delta(t)] = \int_{0}^{\infty} g(\tau) \delta(t-\tau) d\tau \]

\[ = \begin{cases} 
  g(t) & \text{if } t \geq 0 \\
  0 & \text{if } t < 0 
\end{cases} \]

\[ = g(t) \ast u(t) \]
c) No, the system has memory because \( g(t) > 0 \) for \( t > 0 \)

d) Yes, because \( h(t) = 0 \) for all \( t < 0 \)
Problem 3. (30pt) Sinusoidal Inputs to LTI Systems
Consider the discrete-time LTI system

\[ y_n = h_n \ast x_n \]

where \( h_n \) is the real-valued impulse response of the system.

a) Show that if \( x_n = e^{j\omega n} \) then the output has the form \( y_n = Ce^{j\omega n} \) were \( C \) is a complex number.

b) Calculate an expression for \( C \).

\[ a) \quad y_n = h_n \ast x_n \]
\[ = x_n \ast h_n \]
\[ = \sum_{k=-\infty}^{\infty} x_{n-k} h_k \]
\[ = \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)} h_k \]
\[ = \sum_{k=-\infty}^{\infty} e^{j\omega n} e^{-j\omega k} h_k \]
\[ = e^{j\omega n} \sum_{k=-\infty}^{\infty} h_k e^{-j\omega k} \]
\[ = C e^{j\omega n} \]

\[ b) \quad C = \sum_{k=-\infty}^{\infty} h_k e^{j\omega k} \]
Problem 4. (30pt) Digital to Analog Conversion

Consider a linear system

\[ y(t) = S[x_n] \]

with discrete-time input \( x_n \) and continuous time output \( y(t) \).

Assume there exists a period \( T \) such that if \( y(t) = S[x_n] \) then \( y(t - kT) = S[x_{n-k}] \) for all integers \( k \). Notice that this property is similar to time invariance.

Further define the function

\[ h(t) \triangleq S[\delta_n] \]

as the response of the system to a discrete-time impulse.

a) Determine the response of the system to \( x_n = \delta_{n-k} \), a DT delta function at time \( k \). Justify (prove) your answer.

b) Determine the response of the system to any DT input \( x_n \). Justify (prove) your answer.

c) Explain in words why this system might be useful.

\[ S[\delta_{n-k}] = h(t-kT) \quad \text{by assumed property} \]

\[ X_n = \sum_{\kappa=-\infty}^{\infty} X_{\kappa} \delta(n-\kappa) \]

\[ S[X_n] = S\left[\sum_{\kappa=-\infty}^{\infty} X_{\kappa} \delta(n-\kappa)\right] \]

\[ \text{by linearity} \quad \sum_{\kappa=-\infty}^{\infty} X_{\kappa} S\left[\delta(n-\kappa)\right] \]

\[ y(t) = \sum_{\kappa=-\infty}^{\infty} X_{\kappa} h(t-kT) \]

C) This system can be used to convert DT signals to CT signals. It is a DT to CT converter.