Z-Transform

Recall:

$$e^{st} \longrightarrow \boxed{\begin{array}{c} \text{CT} \\ \text{LTI} \\ h_a(t) \end{array}} \longrightarrow H_a(s)e^{st}$$

$$H_a(s) = \underbrace{\mathcal{L}}_{\text{Laplace Transform}}\{h_a(t)\}$$

· Consider:

$$x[n] = a^n \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ h[n]\end{array}} \longrightarrow y[n] = ?$$

for all n

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, a^{n-k}$$

$$= \left\{ \sum_{k=-\infty}^{\infty} h[k]\, a^{-k} \right\} a^n$$

• Defining Z-Transform (ZT) of $h[n]$:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

$$x[n] = a^n \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h[n]\end{matrix}} \longrightarrow y[n] = \underbrace{H(a)}_{\substack{\text{ZT of } h[n] \\ \text{evaluated at} \\ z=a}}\, a^n$$

$-\infty < n < \infty$

- Examples. "right-sided" sequence

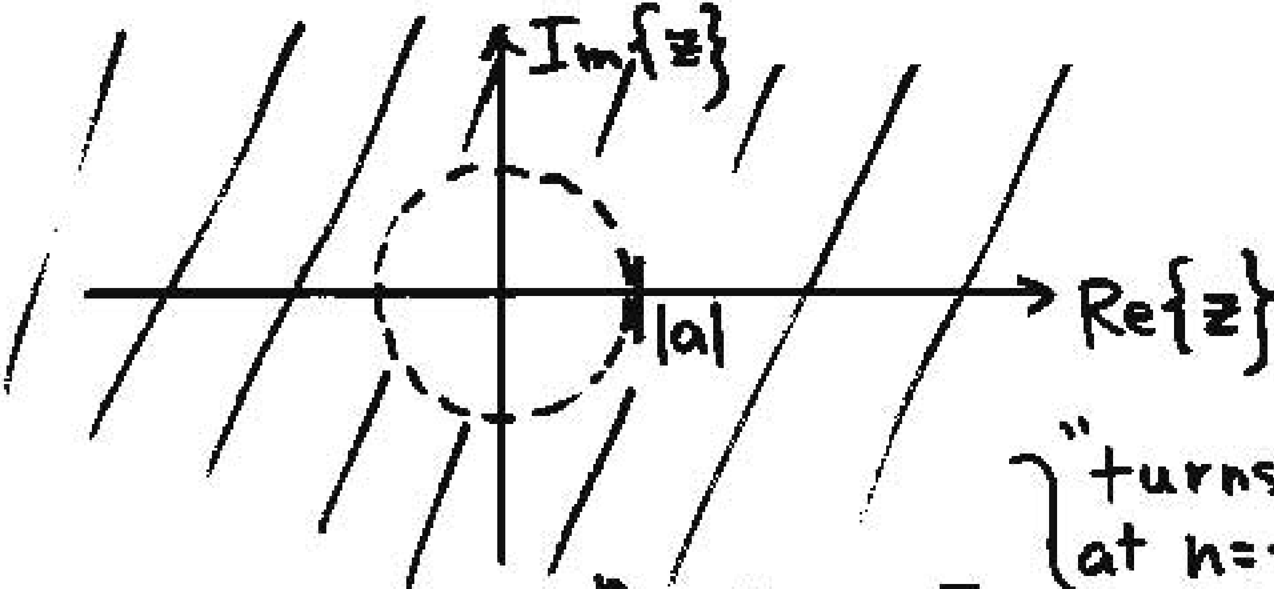
1. $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1-az^{-1}} \qquad |az^{-1}| < 1$$

- Region of Convergence (ROC):
  - any value of z for which $X(z) < \infty$

$$\left|\frac{a}{z}\right| < 1 \Rightarrow |a| < |z| \Rightarrow |z| > |a|$$



2. $x[n] = -a^n u[-n-1]$ } "turns on at $n=-\infty$ and huts off at $n=0$

$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

change of variables:
$n' = -n$
$(n = -n')$

$$X(z) = -\sum_{n'=1}^{\infty} (az^{-1})^{-n'}$$

ROC:

$$= -\sum_{n=1}^{\infty} (a^{-1}z)^{n}$$

$$|a^{-1}z| < 1$$

$$\left|\frac{z}{a}\right| < 1$$

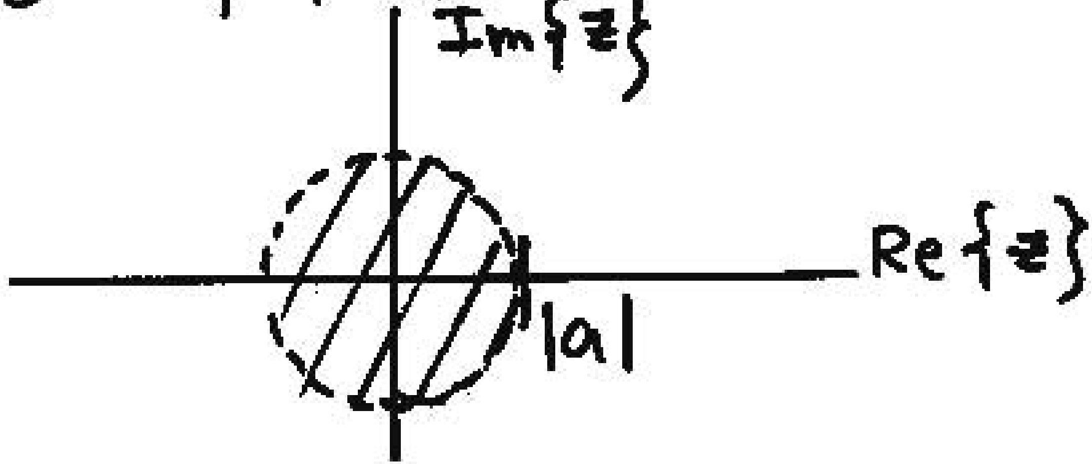
$$|z| < |a|$$

$$= \frac{-1}{1 - a^{-1}z} + 1$$

$$= \frac{-a}{a - z} + 1$$

$$= \frac{a}{z-a} + \frac{z-a}{z-a} = \frac{z}{z-a}$$

ROC: $|z| < |a|$



X(z) is uniquely defined by functional form and ROC

3. $x[n] = a^n u[n] + b^n u[-n-1]$

$\Rightarrow$ 2-sided sequence

$$\mathcal{Z}\{x[n]\} = X(z) = \frac{z}{z-a} - \frac{z}{z-b}$$

ROC: $\{|z| > |a|\} \cap \{|z| < |b|\}$

- if $|b| > |a|$: ROC: $|a| < |z| < |b|$

See pg. 158, Fig. 3.4 $\{$ annular region

- if $|b| < |a|$, $\Rightarrow$ ROC $= \emptyset$

Properties of ZT:

- shifting property:

$$\mathcal{Z}\{x[n-k]\} = z^{-k} X(z)$$

- for $k=1$: $\mathcal{Z}\{x[n-1]\} = z^{-1} X(z)$

$$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow x[n-1]$$

- Convolution Property:

$$\mathcal{Z}\{x[n] * h[n]\} = H(z)X(z)$$

See Table 10.1 on pg. 777 , for further properties of ZT

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h[n]\end{matrix}} \longrightarrow y[n]$$

$$Y(z) = H(z)X(z)$$

$$H(z) = \mathcal{Z}\{h[n]\} = \frac{Y(z)}{X(z)}$$

- z_0 is a pole of $H(z)$ if $H(z_0) = \infty$
- z_0 is a zero of $H(z)$ if $H(z_0) = 0$

• ZT analysis of LTI System described by difference eqns

$$\mathcal{Z}\{y[n]\} = \mathcal{Z}\left\{-\sum_{k=1}^{N} a_k\, y[n-k]\right\} + \mathcal{Z}\left\{\sum_{k=0}^{M} b_k\, x[n-k]\right\}$$

$$Y(z) = -\sum_{k=1}^{N} a_k\, z^{-k}\, Y(z) + \sum_{k=0}^{M} b_k\, z^{-k}\, X(z)$$

shift prop.

• Convolution property:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$= b_0 \frac{z^{-M}}{z^{-N}} \cdot \frac{z^M + \left(\frac{b_1}{b_0}\right) z^{M-1} + \dots + \frac{b_M}{b_0}}{z^N + a_1 z^{N-1} + \dots + a_N}$$

$$= b_0 z^{N-M} \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)}$$

• z_k: zeroes of $H(z)$

• p_k: poles of $H(z)$

- assume poles are unique and $M \leq N$ (if not, perform long division first)

(Keep in mind: $\delta[n-k] \overset{ZT}{\longleftrightarrow} z^{-k}$)

- partial fraction expansion

$$H(z) = A_1 \frac{z}{z-p_1} + A_2 \frac{z}{z-p_2} + \dots + A_N \frac{z}{z-p_N}$$

- where: $A_k = \dfrac{z-p_k}{z} H(z)\Big|_{z=p_k}$

- Basic inversion result:

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-p_k}\right\} = \begin{cases} p_k^n u[n], & \text{if ROC} \subset \{|z|>|p_k|\} \\ -p_k^n u[-n-1], & \text{if ROC} \subset \{|z|<|p_k|\} \end{cases}$$

- for repeated poles, see pp. 191-193 in P+M Text
  $\Rightarrow$ Example 3.4.7

- if $z_i \neq p_j$ $\quad i=1,\dots,M;\ j=1,\dots,N$
  then $H(z)\big|_{z=p_j} = \infty$
- ROC cannot contain a pole
- assume poles ordered as
  $|p_1| \leq |p_2| \leq \dots \leq |p_N|$
- ROC must lie in an annular region: $|p_k| < |z| < |p_{k+1}|$

- **Causality** requires $h[n]=0$ for $n<0$
- implies $h[n]$ must be "right-sided" sequence
- ROC of $H(z)$ must be

$$|z| > |p_N|$$

- p_N: pole with largest magnitude

- Example.

$$y[n] = \frac{13}{4} y[n-1] - \frac{3}{4} y[n-2] + x[n]$$

- Determine all possible impulse responses

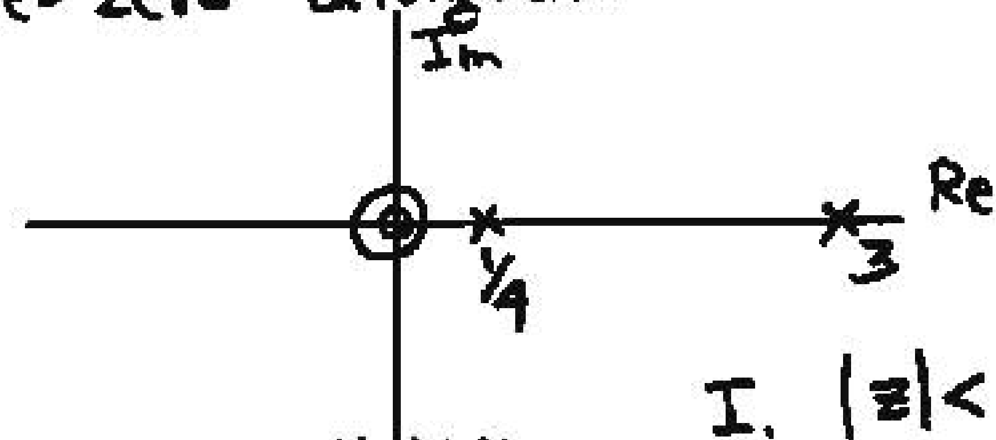
$$Y(z) = \frac{13}{4} z^{-1} Y(z) - \frac{3}{4} z^{-2} Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{13}{4} z^{-1} + \frac{3}{4} z^{-2}}$$

$$H(z) = \frac{z^2}{z^2 - \frac{13}{4} z + \frac{3}{4}} =$$

$$H(z) = \frac{z^2}{\left(z - \frac{1}{4}\right)(z-3)}$$

- pole-zero diagram :



Three possibilities for ROC

I. $|z| < \frac{1}{4}$

II. $\frac{1}{4} < |z| < 3$

III. $|z| > 3$

$$H(z) = A_1 \frac{z}{z-\frac{1}{4}} + A_2 \frac{z}{z-3}$$

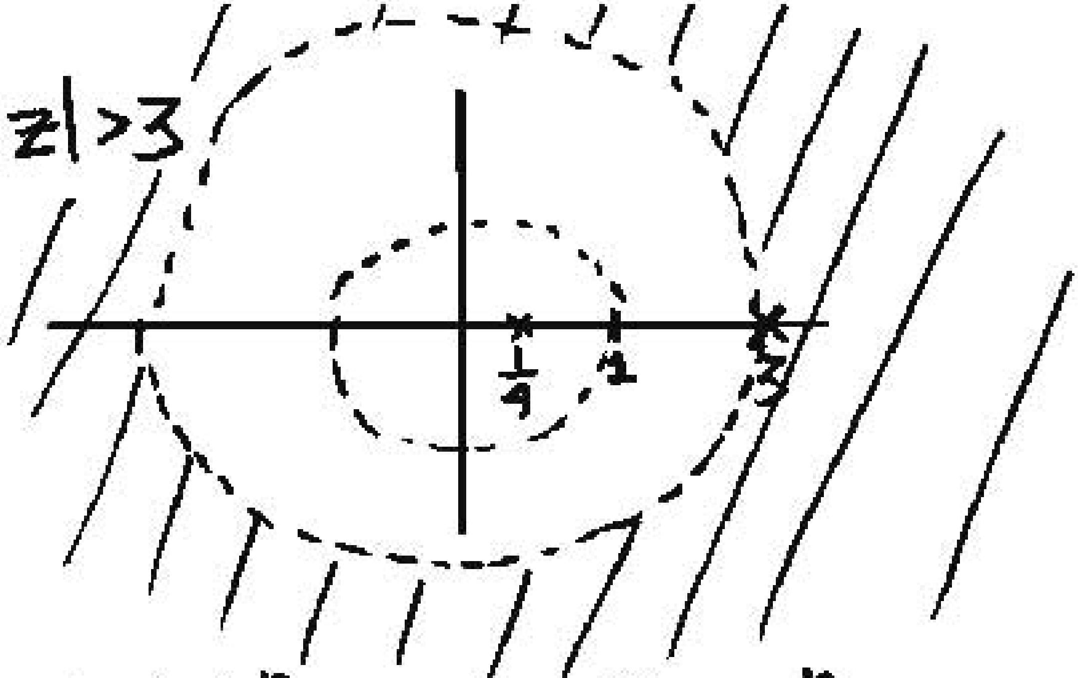
$$A_1 = \frac{z-\frac{1}{4}}{z} H(z)\Big|_{z=\frac{1}{4}} =$$

$$= \frac{z}{z-3}\Big|_{z=\frac{1}{4}} = -\frac{1}{11}$$

$$A_2 = \frac{12}{11}$$

$$H(z) = -\frac{1}{11} \frac{z}{z-\frac{1}{4}} + \frac{12}{11} \frac{z}{z-3}$$

III. $|z|>3$



$$h[n] = -\frac{1}{11}\left(\frac{1}{4}\right)^n u[n] + \frac{12}{11}(3)^n u[n]$$

- Causal? Yes!
- Stable? No!

- <u>Stability</u>: requires ROC to include unit circle, $|z| = 1$
- Supporting argument:
- invoke triangle inequality ($|a+b| \leq |a| + |b|$)
- $$|H(z)| \leq \sum_{n=-\infty}^{\infty} \left| h[n] z^{-n} \right|$$
- on unit circle: $|z^{-n}| = |z|^{-n} = \frac{1}{1^n} = 1$

- on unit circle:

$$\underbrace{|H(z)| \le \sum_{n=-\infty}^{\infty} |h[n]| < \infty}_{\text{for BIBO Stability}}$$

- ROC must include unit circle for BIBO Stability

• Stability and Causality

• requires $|z| > |p_N|$ must include unit circle, $|z| = 1$

$\Rightarrow |p_N| < 1$

$\Rightarrow$ all poles must be located within unit circle

• note: for distinct poles:

$$h[n] = \sum_{k=1}^{N} A_k \, p_k^{\,n} \, u[n]$$

$$x[n] = e^{j\omega_0 n} \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h[n]\end{matrix}} \longrightarrow y[n] = H(\omega_0)\, e^{j\omega_0 n}$$

- where: $H(\omega) \underbrace{=}_{\text{notational problem}} H(z)\Big|_{z=e^{j\omega}}$

- $H(\omega) = \displaystyle\sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$ } DTFT of $h[n]$

- only defined if ROC includes $|z|=1$

  $\Rightarrow$ only defined for stable systems

- Return to Difference Eqns.

$$y[n] = -\sum_{k=1}^{N} a_k \, y[n-k] + \sum_{k=0}^{M} b_k \, x[n-k]$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k \, z^{-k}}{1 + \sum_{k=1}^{N} a_k \, z^{-k}}$$

$$= b_0 \, z^{N-M} \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}$$

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}}$$

$$H(\omega) = b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M} \left(e^{j\omega} - z_k\right)}{\prod_{k=1}^{N} \left(e^{j\omega} - p_k\right)}$$

$$|H(\omega)| = |b_0| \prod_{k=1}^{M} \left|e^{j\omega} - z_k\right| \Big/ \prod_{k=1}^{N} \left|e^{j\omega} - p_k\right|$$

$$\angle H(\omega) = \omega(N-M) + \angle b_0 + \sum_{k=1}^{M} \angle(e^{j\omega} - z_k) - \sum_{k=1}^{N} \angle(e^{j\omega} - p_k)$$