EE 301 Chapter 1

Period 1. Reading assignment: Chapter 1

Distribute/Discuss:
- Role Sheets
- Course Assignment Sheet
- Course Administration Sheet
- Cheating

Course Objective:
Tools for deterministic methods of system and signal processing, both continuous-time and discrete-time.

\[ x(t) \xrightarrow{\text{Linear System}} y(t) \]
\[ x[n] \xrightarrow{} y[n] \]

The Engineering Approach:
Must have:
- Mathematical Representation of the Signals
- Mathematical Representation of the System
- Convenient means for manipulation of these to find solution.

SOLUTION TOOLS TO BE STUDIED

<table>
<thead>
<tr>
<th>System Type</th>
<th>Time Domain</th>
<th>Frequency Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous Time Signals &amp; Systems</td>
<td>Time functions ( x(t) ), Differential Equations, Convolution Integral</td>
<td>Fourier Series, Fourier Transform</td>
</tr>
<tr>
<td>Discrete Time Signals &amp; Systems</td>
<td>Time series ( x[n] ), Difference Equations, Convolution Sum</td>
<td>Discrete Fourier Transform, Z Transform</td>
</tr>
</tbody>
</table>

Differential/Difference Equation: The most basic means for system representation.

<table>
<thead>
<tr>
<th>Nth order</th>
<th>Ordinary</th>
<th>Linear</th>
<th>Differential Eq.</th>
<th>Non-Constant Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>Definition</td>
<td>Order of Highest Derivative</td>
<td>Only One Independent Variable</td>
<td>Coefficients not a function of dependent variable</td>
<td>Terms contain only derivatives</td>
</tr>
<tr>
<td>Alternative</td>
<td>Mth order</td>
<td>partial</td>
<td>nonlinear</td>
<td>difference eq.</td>
</tr>
<tr>
<td>System Implication</td>
<td>System Complexity</td>
<td>lumped vs distributed</td>
<td>superposition apply?</td>
<td>continuous vs discrete time</td>
</tr>
</tbody>
</table>
Period 2. Reading assignment: §2.0-2.2

A further comment, beyond that in the text, on the practicality of discrete time representations of continuous time signals:

Making a measurement of a voltage implies quantization, e.g. if one reads a voltmeter as 3.2 volts \( \Rightarrow 3.15 < v < 3.25 \) volts. Discuss the relationship between quantization, error, and uncertainty.

Reading a time-varying voltage at a specific time implies the same type of discretization but on the time axis, thus,

\[
v(t)
\]

Difference equations were mentioned as a basic means for mathematically representing discrete time systems. As a result of the above discretization, the calculus needed is difference calculus, not differential calculus, i.e. don't pass to the limit of \( \Delta t \to 0 \). Thus the following approximation is used.

\[
\frac{dy}{dt} \approx \frac{y(t+\Delta t) - y(t)}{\Delta t}
\]

Thus a first order differential equation such as,

\[
a_1 \frac{dy}{dt} + a_0 y(t) = b_0 x(t)
\]

becomes,

\[
a_1 \frac{y(t+\Delta t) - y(t)}{\Delta t} + a_0 y = b_0 x
\]

But we can only evaluate it at discrete times \( n\Delta t \), not all times \( t \). Thus,

\[
a_1 y((n+1)\Delta t) = a_1 y(n\Delta t) - a_0 y(n\Delta t)\Delta t + b_0 x(n\Delta t)\Delta t
\]
Suppressing the $\Delta t$'s, and adopting the author's notation using $[]$, this becomes,

$$y[n+1] = y[n] - a_0'y[n] + b'x[n]$$

or

$$y[n+1] = a_0''y[n] + b'x[n]$$

where,

$$a_0'' = 1 - a_0'$$

$$= 1 - \frac{a_0\Delta t}{a_1}$$

and

$$b' = \frac{b\Delta t}{a_1}$$

Note that this solution is in the form of a recursion relation and can thus easily be solved.

This perhaps conveys the basic idea of the difference equation compared to the differential equation.

Next, the author talks about some transformations and special signals, specifically,

<table>
<thead>
<tr>
<th>Continuous Time</th>
<th>Discrete Time</th>
<th>Trans./Fct. Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(t) \rightarrow x(-t)$</td>
<td>$x[n] \rightarrow x[-n]$</td>
<td>Flopping (about ordinate)</td>
</tr>
<tr>
<td>$x(t) \rightarrow x(at)$</td>
<td>$x[n] \rightarrow x[n]$ (new interval)</td>
<td>Scaling</td>
</tr>
<tr>
<td>$x(t) \rightarrow x(t - t_0)$</td>
<td>$x[n] \rightarrow x[n-n_0]$</td>
<td>Shifting</td>
</tr>
<tr>
<td>$x(t) = x(-t)$</td>
<td>$x[n] = x[-n]$</td>
<td>Even Function</td>
</tr>
<tr>
<td>$x(t) = -x(-t)$</td>
<td>$x[n] = -x[-n]$</td>
<td>Odd Function</td>
</tr>
<tr>
<td>$x(t) = x(t+T)$ for all t</td>
<td>$x[n] = x[n+N]$ for all n</td>
<td>Periodic Function</td>
</tr>
</tbody>
</table>

Now suppose we decompose an arbitrary signal into its component symmetrical parts:

$$x(t) = x_e(t) + x_o(t)$$

then from the definitions,

$$x(-t) = x_e(t) - x_o(t)$$

Adding, we get,

$$x(t) + x(-t) = 2x_e(t)$$

or,

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

Note that $x_e(t)$ is indeed an even function. In a similar fashion we obtain,

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$
Again, note that $x_0(t)$ is indeed an odd function. Thus, we have shown that it is possible to separate an arbitrary function into even and odd parts, and we have also determined how to calculate these symmetrical parts.

Now do some graphical examples of $x(at)$, $x(-t)$, $x(t-a)$, symmetry and periodicity.

Use of shifted functions is a useful way to write an equation for such geometric functions. For example, for the ramp portion of $x(t)$, start with a ramp function at $t=0$, then shift it one unit to the right. The equation for the whole function can be written as,

$$x(t) = (t-1)u(t-1) - (t-2)u(t-2) - u(t-3)$$

check:

at $t = 1.5$: $x(1.5) = 0.5(1) - (0.5)(0) - 0 = 0.5$

at $t = 2.5$: $x(2.5) = 1.5(1) - (0.5)(1) - 0 = 1$

at $t = 4$: $x(4) = 3(1) - 2(1) - 1 = 0$

Period 3. Reading Assignment: §2.3-2.4

Basic Continuous-Time Signals

$$x(t) = Ce^{at}$$
Signal form is familiar if \( a \) is real. If \( a \) is imaginary, the signal is complex:

\[
\Re^{j\omega t} = \Re^{j\omega(t + T)} = \Re^{j\omega t} \Re^{j\omega T}
\]

Thus, it is periodic if

\[
\Re^{j\omega T} = 1
\]

Or,

\[
T = \frac{2\pi}{|\omega|}
\]

An important and useful relation involving this function is Euler's Relation

\[
\Re^{j\omega t} = \cos \omega t + j \sin \omega t
\]

It is useful in converting from polar to cartesian form. Note that,

\[
M \Re^{j\theta} = M \cos \theta + j M \sin \theta = a + jb
\]

Operations like addition and subtraction of complex numbers best done in cartesian form while multiplication, division, powers and roots must be done in polar form. Thus,

Problem 2.5 (a)

\[
\frac{3 + 4j}{1 - 2j} = ?
\]

Or,

\[
\frac{6}{\sqrt{1 + j}} = ?
\]

require that one make use of this relationship.

The Unit Step Function,

\[
u(t) = \begin{cases} 
0 & t < 0 \\
1 & t > 0 
\end{cases}
\]
Note that $u(t)$ is not defined at $t = 0$ in this case. Note also that its derivative is zero everywhere except at $t = 0$ where it also is not defined. Conceptually, both of these conditions can be overcome by using a limiting process on the function $u_\Delta(t)$,

\[
\begin{align*}
\lim_{\Delta \to 0} u_\Delta(t) = \delta(t)
\end{align*}
\]

Note that,

\[
\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(\tau)
\]

and,

\[
\int_{-\infty}^{t} k\delta(\tau) \, d\tau = k \, u(t)
\]

Also,

\[
\tilde{f}(t) \, \delta(t-a) = \tilde{f}(a) \, \delta(t-a)
\]

**DISCRETE-TIME SIGNALS**

Unit Step sequence,

\[
u[n] = \begin{cases} 
0 & n < 0 \\
1 & n \geq 0 
\end{cases}
\]

Unit Pulse (unit sample)

\[
\delta[n] = \begin{cases} 
0 & n \neq 0 \\
1 & n = 0 
\end{cases}
\]

Note that,

\[
x[n] \, \delta[n] = x[0] \, \delta[n]
\]

Analogies:

<table>
<thead>
<tr>
<th>CONTINUOUS</th>
<th>DISCRETE</th>
<th>DISCRETE RELATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derivative</td>
<td>First Difference</td>
<td>$\delta[n] = u[n] - u[n-1]$</td>
</tr>
<tr>
<td>Integral</td>
<td>Sum</td>
<td>$u[n] = \sum_{m=-\infty}^{n} \delta[m]$</td>
</tr>
</tbody>
</table>
DISCRETE TIME COMPLEX EXPONENTIAL

\[ x[n] = C e^{\beta n} = C \alpha^n \quad \text{where} \quad \alpha = e^{\beta} \]

If \( \alpha \) is real, \( x[n] \) can:
- grow with \( n \) \( \alpha > 1 \)
- decline with \( n \) \( 0 < \alpha < 1 \)
- alternate and decline with \( n \) \( -1 < \alpha < 0 \)
or,
- alternate and grow with \( n \) \( \alpha < -1 \)

For the case \( \beta \) imaginary,

\[ x[n] = e^{i\Omega n} \]

Recall that the continuous version of the complex exponential has the following two properties,

a) \( e^{i\omega t} \) is periodic for any value of \( \omega \), and
b) the larger \( \omega \) the higher the periodicity rate.

Neither of these are the true in the discrete case as we will see.

First, note that

\[ e^{i(\Omega+2\pi)n} = e^{i\Omega n} e^{i2\pi n} = e^{i\Omega n} \]

Thus, an exponential at frequency \( \Omega + 2\pi \) is the same as one at \( \Omega \).
- In the continuous case, functions of \( e^{i\omega t} \) are distinct for all values of \( \omega \),
- In the discrete case, they are only distinct over a range of \( 2\pi \).
The oscillation increases over the first half of a \( 2\pi \) range, then decreases again.

Next note that for \( e^{i\Omega n} \) to be periodic with period \( N \), we must have

\[ e^{i\Omega(n+N)} = e^{i\Omega n} \]
or,

\[ e^{i\Omega N} = 1 \]

Thus,

\[ \Omega N = 2\pi m \]
or,

\[ \frac{\Omega}{2\pi} = \frac{m}{N} \]

Thus \( e^{i\Omega n} \) is periodic only if \( \frac{\Omega}{2\pi} \) is a rational number.
Period 4. Reading assignment: §2.5-2.7

A System is any process that results in the transformation of signals.

Systems may be interconnected in

- Series or cascade,
- Parallel, or
- Complex combinations of these including feedback connections.

Properties of Systems

- **With/Without Memory** – Output at each time is dependent only on the input at that time.

  \[
  y(t) = \frac{R_2}{R_1 + R_2} x(t)
  \]

- **Invertibility** and **Inverse Systems** – If by observing the outputs we can determine the inputs.

  \[
  x(t) \Rightarrow y(t) \Rightarrow x(t)
  \]

- **Causality** or **Non-anticipatory** – Output at any time depends only on inputs at the present and past times.

  It is desirable but not physically realizable to build predictor (anticipatory) systems, however, in some circumstances, where the data are recorded for later processing, it is possible to use algorithms which are non-causal.

- **Stability** – Produces a bounded output for every bounded input.
• **Time Invariance** – A time-shift in input causes a corresponding time shift in output but does not otherwise change the output.

\[
\frac{dy}{dt} + \frac{1}{R(t)C} y = \frac{1}{R(t)C} x
\]

• **Linearity** – Superposition applies.

If,

- \( x_1(t) \Rightarrow y_1(t) \) and
- \( x_2(t) \Rightarrow y_2(t) \), then
- \( ax_1(t) + bx_2(t) \Rightarrow ay_1(t) + by_2(t) \)

\[
\frac{dy}{dt} + \frac{1}{R(i)C} y = \frac{1}{R(i)C} x \quad \text{or}
\]

\[
\frac{dy}{dt} + \frac{1}{R(x,y)C} y = \frac{1}{R(x,y)C} x
\]