Summary of VIP Information for Fourier Series (Continuous Time)

Chapter 3

$$\phi_k(t) = e^{j 2\pi \frac{k}{T} t}, \quad k, \text{ integer} \quad -\infty < k < \infty$$

form a complete and orthogonal basis for all periodic functions with period T

Orthogonality implies: $\displaystyle\int_{t_0}^{t_0+T} \phi_k(t)\, \phi_\ell^*(t)\, dt =$

$$= \int_{t_0}^{t_0+T} e^{j 2\pi \frac{k}{T} t} e^{-j 2\pi \frac{\ell}{T} t}\, dt = \begin{cases} T, & \text{if } k = \ell \\ 0, & \text{if } k \neq \ell \end{cases}$$

$$\boxed{\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} c_k\, e^{j 2\pi \frac{k}{T} t} \\ c_k &= \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, e^{-j 2\pi \frac{k}{T} t}\, dt \end{aligned}}$$

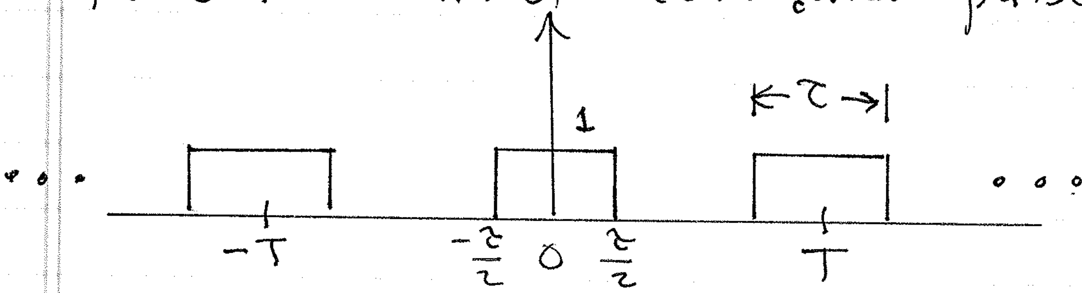
Basic Fourier Series

- Periodic train of Dirac Delta functions

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{+j 2\pi \frac{k}{T} t} \qquad c_k = \frac{1}{T} \text{ for all } k$$

- Periodic train of rectangular pulses



$$x(t) = \sum_{n=-\infty}^{\infty} \operatorname{rect}\left(\frac{t - nT}{\tau}\right)$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j 2\pi \frac{k}{T} t}$$

$$c_k = \frac{\sin\left(k\pi \frac{\tau}{T}\right)}{k\pi}$$

Some Key Fourier Series Properties

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j 2\pi \frac{k}{T} t}}$$

- **Time Shift:**

$$y(t) = x(t - t_0) = \sum_{k=-\infty}^{\infty} c_k e^{j 2\pi \frac{k}{T}(t - t_0)}$$

$$= \sum_{k=-\infty}^{\infty} \left(c_k e^{-j 2\pi \frac{k}{T} t_0} \right) e^{j 2\pi \frac{k}{T} t}$$

- **Differentiation:**

$$y(t) = \frac{d\,x(t)}{dt} = \sum_{k=-\infty}^{\infty} \left(j 2\pi \frac{k}{T} c_k \right) e^{j 2\pi \frac{k}{T} t}$$

Time-Scaling:

$$y(t) = x(at) = \sum_{k=-\infty}^{\infty} c_k e^{j 2\pi \frac{k}{T} a t}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j 2\pi \frac{k}{T/a} t}$$

F.S. coefficients unchanged $\Rightarrow$ new period $= \frac{T}{a}$

Parseval's Theorem for FS:

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Proof: Substitute $x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$

$$x^*(t) = \sum_{l=-\infty}^{\infty} a_l^* \phi_l^*(t)$$

$$\phi_k(t) = e^{j\frac{2\pi k}{T}t}$$

Recall: $\int_{-T/2}^{T/2} \phi_k(t)\phi_l^*(t)\, dt = T\delta[k-l]$

$= T$ if $k=l$

0 if $k \neq l$

Back to proof:

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x^*(t)\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} a_k \phi_k(t) \sum_{l=-\infty}^{\infty} a_l^* \phi_l^*(t)\, dt$$

$$= \sum_k \sum_l a_k a_l^* \underbrace{\frac{1}{T}\int_{-T/2}^{T/2} \phi_k(t)\phi_l^*(t)\, dt}_{=0 \text{ if } k\neq l \qquad =1 \text{ if } k=l}$$

$$= \sum_{k=-\infty}^{\infty} |a_k|^2$$