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Inverse 2D-FT in polar coordinates:

$$\tilde{f}(r,\theta) = \int_0^{2\pi}\int_0^{\infty} \tilde{F}(\rho,\phi)\, e^{j2\pi\rho r\cos(\phi-\theta)}\, \rho\, d\rho\, d\phi$$

2D-FT is (in polar coordinates)

$$\tilde{F}(\rho,\phi) = \int_0^{2\pi}\int_0^{\infty} \tilde{f}(r,\theta)\, e^{-j2\pi\rho r\cos(\phi-\theta)}\, r\, dr\, d\theta$$

<u>Note:</u>

$$\tilde{F}(\rho,\phi+\pi) = \int_0^{2\pi}\int_0^{\infty} \tilde{f}(r,\theta)\, e^{-j2\pi\rho r\cos(\phi+\pi-\theta)}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\infty} \tilde{f}(r,\theta)\, e^{-j2\pi(-\rho) r\cos(\phi-\theta)}\, r\, dr\, d\theta$$

$$\tilde{F}(\rho,\phi+\pi) = \tilde{F}(-\rho,\phi)$$ } property of 2D-FT

- All the ensuing math is to write the Inverse 2D-FT so that the limits for ϕ are $0 < \phi < \pi$

phi

- **Final answer:**

$$\tilde{f}(r,\theta) = \int_0^{\pi} \int_{-\infty}^{\infty} \tilde{F}(\rho,\phi)\, e^{j 2\pi \rho r \cos(\phi-\theta)}\, |\rho|\, d\rho\, d\phi$$

- **First step:**

$$\tilde{f}(r,\theta) = \int_0^{2\pi} \int_0^{\infty} \{\text{integrand above}\}\, \rho\, d\rho\, d\phi$$

$$= \left.\int_0^{\pi} \int_0^{\infty} \{\text{integrand above}\}\, \rho\, d\rho\, d\phi \right\} \text{Integral (A)}$$

$$+ \left.\int_{\pi}^{2\pi} \int_0^{\infty} \{\text{integrand above}\}\, \rho\, d\rho\, d\phi \right\} \text{Integral (B)}$$

• Work on B so limits for ϕ are $\int_0^{\pi}$

• Step A. change of variables $\phi' = \phi - \pi$

B $\quad \phi = \phi' + \pi \qquad \phi' \Big]_0^{\pi} \qquad d\phi' = d\phi$

$$\int_0^{\pi}\int_0^{\infty} \tilde{F}(\rho, \phi' + \pi)\, e^{j 2\pi \rho r \cos(\phi' + \pi - \theta)}\, \rho\, d\rho\, d\phi$$

drop prime'

$$= \int_0^{\pi}\int_0^{\infty} \tilde{F}(-\rho, \phi)\, e^{j 2\pi (-\rho) \cos(\phi - \theta)}\, \rho\, d\rho\, d\phi$$

change of variables: $\rho' = -\rho \qquad \rho' \Big]_0^{-\infty} \qquad d\rho' = -d\rho$

$$= \int_0^{\pi}\int_0^{-\infty} \tilde{F}(\rho', \phi)\, e^{j 2\pi \rho' \cos(\phi - \theta)}\, (-\rho')(-d\rho')\, d\phi$$

cancel

drop primes'

change/reverse limits on integration $\int_a^b = -\int_b^a$ (4)

$$B = -\int_0^{\pi}\int_{-\infty}^{0} \tilde{F}(\rho,\phi)\, e^{j2\pi\rho\cos(\phi-\theta)}\, \rho\, d\rho\, d\phi$$

$$= \int_0^{\pi}\int_{-\infty}^{0} \tilde{F}(\rho,\phi)\, e^{j2\pi\rho\cos(\phi-\theta)}\, (-\rho)\, d\rho\, d\phi$$

over: $-\infty < \rho < 0$, $-\rho = |\rho|$

$$B = \int_0^{\pi}\int_{-\infty}^{0} \tilde{F}(\rho,\phi)\, e^{j2\pi\rho\cos(\phi-\theta)}\, |\rho|\, d\rho\, d\phi$$

A + B

$$\tilde{f}(r,\theta) = \int_0^{\pi}\int_{-\infty}^{\infty} \tilde{F}(\rho,\phi)\, e^{j2\pi\rho r\cos(\phi-\theta)}\, |\rho|\, d\rho\, d\phi$$

$$r\cos(\phi-\theta) = (\cos\phi\cos\theta + \sin\phi\sin\theta)r$$
$$= \cos\phi \, r\cos\theta + \sin\phi \, r\sin\theta$$
$$= x\cos\phi + y\sin\phi$$

• Substitute:

$$\underbrace{f(x,y)}_{\substack{\text{image}\\\text{in rect}\\\text{coordinates}}} = \int_0^{\pi} \int_{-\infty}^{\infty} \underbrace{\tilde{F}(\rho,\phi)}_{\substack{\text{2D-FT}\\\text{in polar}\\\text{coordinates}}} e^{j2\pi\rho(x\cos\phi + y\sin\phi)} |\rho| \, d\rho \, d\phi$$

• Now, discretize ϕ: $\phi_k = k\frac{\pi}{N}$ $k = 0, 1, \ldots, N$

$$f(x,y) \approx \sum_{k=0}^{N-1} \underbrace{\frac{\pi}{N}}_{\Delta\phi_k} \int_{-\infty}^{\infty} \tilde{F}(\rho,\phi_k) e^{j2\pi\rho(x\cos\phi_k + y\sin\phi_k)} |\rho| \, d\rho$$

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- Rearranging:

$$f(x,y) = \frac{\pi}{N} \sum_{k=0}^{N-1} \int_{-\infty}^{\infty} \tilde{F}(\rho, \phi_k)\, |\rho|\, e^{j2\pi\rho t}\, d\rho$$

where: $t = x\cos\phi_k + y\sin\phi_k$ is a line

as t varies, $x\cos\phi_k + y\sin\phi_k = t$

describes a sequence of lines all of which

are parallel with diffent y-axis intercepts

- Note:

$\tilde{F}(\rho, \phi_k)$ is a 1-D slice of the 2D FT

at an angle of $\phi_k = \theta_k$

$$\tilde{F}(\rho, \phi_k) = \mathcal{F}\{P_{\phi_k}(t)\}$$

Filtered Back-Projection Algorithm

For $k = 0, 1, \ldots, N-1$:

1. $\tilde{F}(\rho, \phi_k) = \mathcal{F}\{p_{\phi_k}(t)\}$

2. Filter: multiply in frequency domain

$$Q_{\phi_k}(\rho) = \tilde{F}(\rho, \phi_k)\,|\rho|$$

3. Compute Inverse 1D-FT of $Q_{\phi_k}(\rho)$ rho

$$q_{\phi_k}(t) = \int_{-\infty}^{\infty} Q_{\phi_k}(\rho)\, e^{j2\pi\rho t}\, d\rho$$

4. Back-projection:

$$g_k(x, y) = q_{\phi_k}(x\cos\phi_k + y\sin\phi_k)$$

5. $f_{k+1}(x, y) = f_k(x, y) + g_k(x, y)$