Sampling a sinewave $x_a(t) = e^{j\omega_a t}$

$$x[n] = x_a(nT_s) = e^{j\omega_a n T_s}$$

$$= e^{j\omega_d n} \qquad \text{where} \quad \omega_d = \omega_a T_s = \frac{\omega_a}{F_s}$$

Now: $$e^{j\omega_a t} \overset{\mathcal{F}}{\underset{\text{CTFT}}{\longleftrightarrow}} X_a(\omega) = 2\pi\,\delta(\omega - \omega_a)$$

Thus: $$X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} 2\pi\,\delta\left(\omega - \omega_a - k\frac{2\pi}{T_s}\right)$$

Sidenote: $\delta(ax) = \frac{1}{|a|}\delta(x)$

$$\int_{-\infty}^{\infty} \delta(ax)\,dx \Rightarrow$$ change of variables

$u = ax \qquad du = a\,dx$

same limits

just consider case of $a>0$

$$\int_{-\infty}^{\infty} \delta(u)\,\frac{1}{a}\,du = \frac{1}{a}$$

So, area under $\delta(ax)$ is $\frac{1}{a}$

Since both $\delta(x)$ and $\delta(ax)$ are only nonzero at $x=0$

can conclude $\delta(ax) = \frac{1}{a}\delta(x)$

THUS: $$X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} 2\pi\,\delta\left(\frac{1}{T_s}\left(\omega T_s - \omega_a T_s - k2\pi\right)\right)$$

$$= \frac{1}{\cancel{T_s}} \sum_k 2\pi \frac{\cancel{T_s}}{1}\,\delta\left(\omega T_s - \omega_a T_s - k2\pi\right)$$

$$X_s(\omega) = \sum_k 2\pi\delta(\omega T_s - \omega_a T_s - k2\pi)$$

Finally, $X(\omega) = X_s(F_s\omega) = X_s\left(\frac{\omega}{T_s}\right)$

$$= \sum_k 2\pi\delta\left(\frac{\omega}{T_s}\cdot T_s - \omega_a T_s - k2\pi\right)$$

$\omega_d = \omega_a T_s$
$= \frac{\omega_a}{F_s}$

$$= \sum_k 2\pi\delta(\omega - \omega_d - k2\pi)$$

<u>VIP</u>: There is <u>no</u> amplitude scaling by the sampling rate

Verifies entry in Table 5.2 for DT sinewave

$$e^{j\omega_0 n} \xleftrightarrow{DTFT} \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 - k2\pi)$$

Dirac Delta function at ω_0 and every integer multiple of 2π away from ω_0
$\Rightarrow$ DTFT is periodic with period 2π

See Figs. 7.15 and 7.16 relative to sampling a sinewave

<u>Note</u>: ω_M = max freq. for a sinewave with analog frequency ω_0 is ω_0 $\Rightarrow$ Nyquist rate is twice the frequency of the sinewave