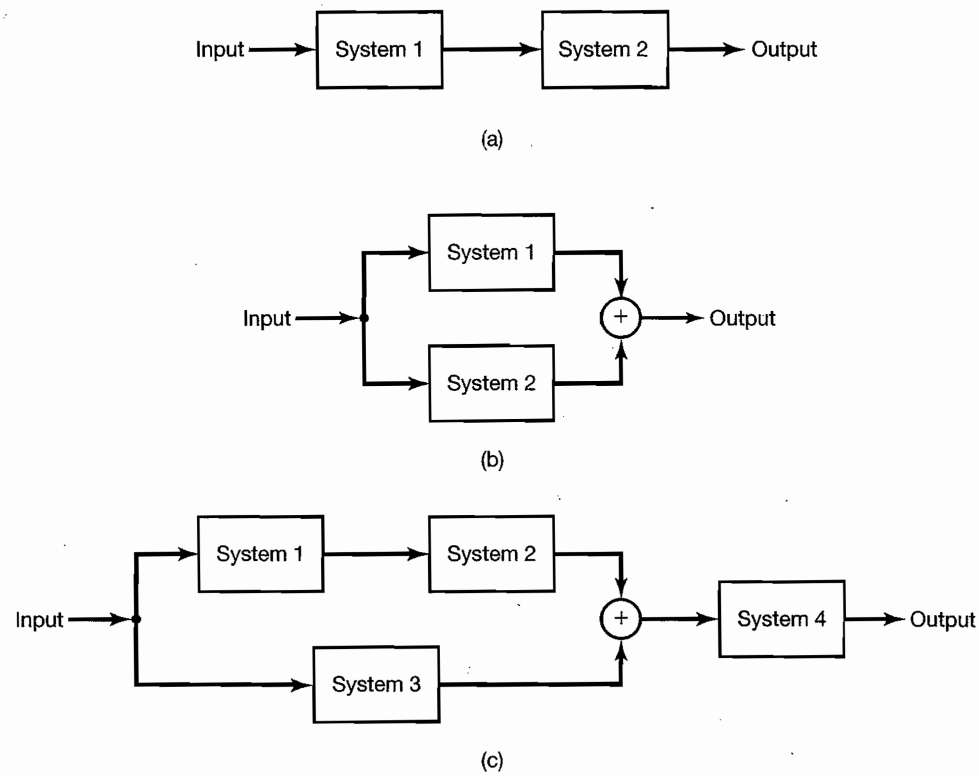



**Figure 1.42** Interconnection of two systems: (a) series (cascade) interconnection; (b) parallel interconnection; (c) series-parallel interconnection.

systems and of how they are interconnected in order to analyze the operation and behavior of the overall system. In addition, by describing a system in terms of an interconnection of simpler subsystems, we may in fact be able to define useful ways in which to synthesize complex systems out of simpler, basic building blocks.

While one can construct a variety of system interconnections, there are several basic ones that are frequently encountered. A *series* or *cascade interconnection* of two systems is illustrated in Figure 1.42(a). Diagrams such as this are referred to as *block diagrams.* Here, the output of System 1 is the input to System 2, and the overall system transforms an input by processing it first by System 1 and then by System 2. An example of a series interconnection is a radio receiver followed by an amplifier. Similarly, one can define a series interconnection of three or more systems.

A *parallel interconnection* of two systems is illustrated in Figure 1.42(b). Here, the same input signal is applied to Systems 1 and 2. The symbol "⊕" in the figure denotes addition, so that the output of the parallel interconnection is the sum of the outputs of Systems 1 and 2. An example of a parallel interconnection is a simple audio system with several microphones feeding into a single amplifier and speaker system. In addition to the simple parallel interconnection in Figure 1.42(b), we can define parallel interconnections of more than two systems, and we can combine both cascade and parallel interconnections

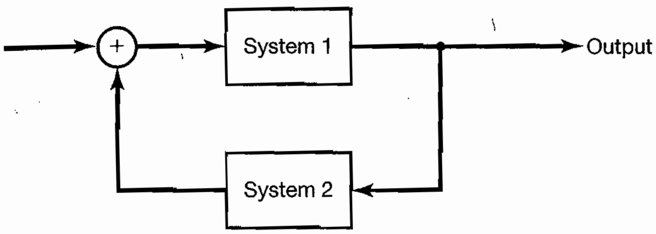
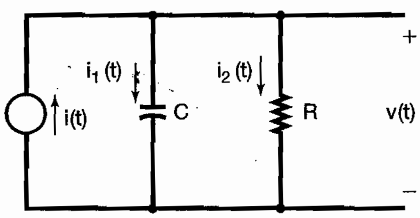



**Figure 1.43** Feedback interconnection.

to obtain more complicated interconnections. An example of such an interconnection is given in Figure 1.42(c).[4]

Another important type of system interconnection is a *feedback interconnection*, an example of which is illustrated in Figure 1.43. Here, the output of System 1 is the input to System 2, while the output of System 2 is fed back and added to the external input to produce the actual input to System 1. Feedback systems arise in a wide variety of applications. For example, a cruise control system on an automobile senses the vehicle's velocity and adjusts the fuel flow in order to keep the speed at the desired level. Similarly, a digitally controlled aircraft is most naturally thought of as a feedback system in which differences between actual and desired speed, heading, or altitude are fed back through the autopilot in order to correct these discrepancies. Also, electrical circuits are often usefully viewed as containing feedback interconnections. As an example, consider the circuit depicted in Figure 1.44(a). As indicated in Figure 1.44(b), this system can be viewed as the feedback interconnection of the two circuit elements.



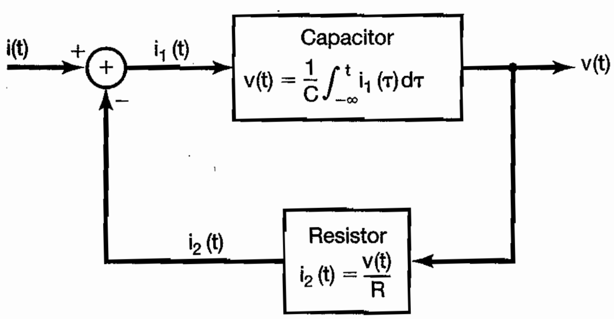




**Figure 1.44** (a) Simple electrical circuit; (b) block diagram in which the circuit is depicted as the feedback interconnection of two circuit elements.

[4]On occasion, we will also use the symbol ⊗ in our pictorial representation of systems to denote the operation of multiplying two signals (see, for example, Figure 4.26).

## 1.6 BASIC SYSTEM PROPERTIES

In this section we introduce and discuss a number of basic properties of continuous-time and discrete-time systems. These properties have important physical interpretations and relatively simple mathematical descriptions using the signals and systems language that we have begun to develop.

### 1.6.1 Systems with and without Memory

A system is said to be *memoryless* if its output for each value of the independent variable at a given time is dependent only on the input at that same time. For example, the system specified by the relationship

$$y[n] = (2x[n] - x^2[n])^2 \tag{1.90}$$

is memoryless, as the value of $y[n]$ at any particular time $n_0$ depends only on the value of $x[n]$ at that time. Similarly, a resistor is a memoryless system; with the input $x(t)$ taken as the current and with the voltage taken as the output $y(t)$, the input-output relationship of a resistor is

$$y(t) = Rx(t), \tag{1.91}$$

where $R$ is the resistance. One particularly simple memoryless system is the *identity system,* whose output is identical to its input. That is, the input-output relationship for the continuous-time identity system is

$$y(t) = x(t),$$

and the corresponding relationship in discrete time is

$$y[n] = x[n].$$

An example of a discrete-time system with memory is an *accumulator* or *summer*

$$y[n] = \sum_{k=-\infty}^{n} x[k], \tag{1.92}$$

and a second example is a *delay*

$$y[n] = x[n-1]. \tag{1.93}$$

A capacitor is an example of a continuous-time system with memory, since if the input is taken to be the current and the output is the voltage, then

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau, \tag{1.94}$$

where $C$ is the capacitance.

Roughly speaking, the concept of memory in a system corresponds to the presence of a mechanism in the system that retains or stores information about input values at times other than the current time. For example, the delay in eq. (1.93) must retain or store the preceding value of the input. Similarly, the accumulator in eq. (1.92) must "remember" or store information about past inputs. In particular, the accumulator computes the running sum of all inputs up to the current time, and thus, at each instant of time, the accumulator must add the current input value to the preceding value of the running sum. In other words, the relationship between the input and output of an accumulator can be described as

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n], \tag{1.95}$$

or equivalently,

$$y[n] = y[n-1] + x[n]. \tag{1.96}$$

Represented in the latter way, to obtain the output at the current time $n$, the accumulator must remember the running sum of previous input values, which is exactly the preceding value of the accumulator output.

In many physical systems, memory is directly associated with the storage of energy. For example, the capacitor in eq. (1.94) stores energy by accumulating electrical charge, represented as the integral of the current. Thus, the simple $RC$ circuit in Example 1.8 and Figure 1.1 has memory physically stored in the capacitor. Similarly, the automobile in Figure 1.2 has memory stored in its kinetic energy. In discrete-time systems implemented with computers or digital microprocessors, memory is typically directly associated with storage registers that retain values between clock pulses.

While the concept of memory in a system would typically suggest storing *past* input and output values, our formal definition also leads to our referring to a system as having memory if the current output is dependent on *future* values of the input and output. While systems having this dependence on future values might at first seem unnatural, they in fact form an important class of systems, as we discuss further in Section 1.6.3.

### 1.6.2 Invertibility and Inverse Systems

A system is said to be *invertible* if distinct inputs lead to distinct outputs. As illustrated in Figure 1.45(a) for the discrete-time case, if a system is invertible, then an *inverse system* exists that, when cascaded with the original system, yields an output $w[n]$ equal to the input $x[n]$ to the first system. Thus, the series interconnection in Figure 1.45(a) has an overall input-output relationship which is the same as that for the identity system.

An example of an invertible continuous-time system is

$$y(t) = 2x(t), \tag{1.97}$$

for which the inverse system is

$$w(t) = \frac{1}{2}y(t). \tag{1.98}$$

This example is illustrated in Figure 1.45(b). Another example of an invertible system is the accumulator of eq. (1.92). For this system, the difference between two successive

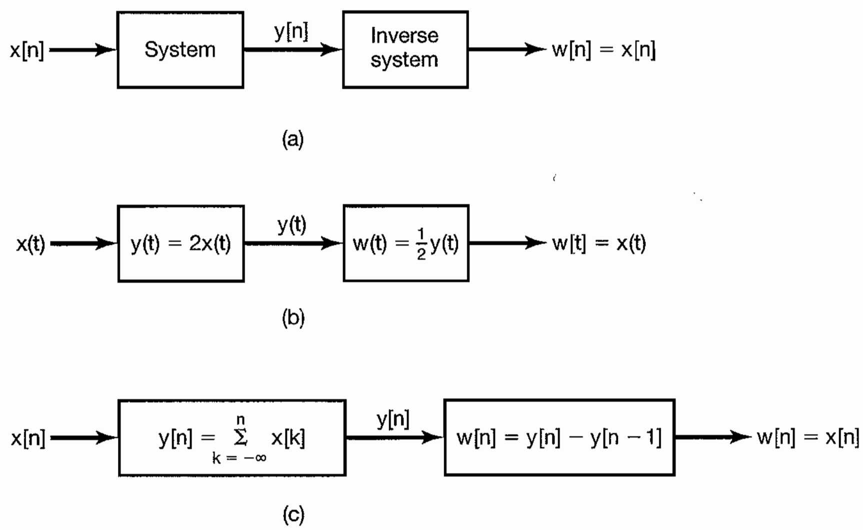



**Figure 1.45** Concept of an inverse system for: (a) a general invertible system; (b) the invertible system described by eq. (1.97); (c) the invertible system defined in eq. (1.92).

values of the output is precisely the last input value. Therefore, in this case, the inverse system is

$$w[n] = y[n] - y[n-1], \tag{1.99}$$

as illustrated in Figure 1.45(c). Examples of noninvertible systems are

$$y[n] = 0, \tag{1.100}$$

that is, the system that produces the zero output sequence for any input sequence, and

$$y(t) = x^2(t), \tag{1.101}$$

in which case we cannot determine the sign of the input from knowledge of the output.

The concept of invertibility is important in many contexts. One example arises in systems for encoding used in a wide variety of communications applications. In such a system, a signal that we wish to transmit is first applied as the input to a system known as an encoder. There are many reasons for doing this, ranging from the desire to encrypt the original message for secure or private communication to the objective of providing some redundancy in the signal (for example, by adding what are known as parity bits) so that any errors that occur in transmission can be detected and, possibly, corrected. For *lossless* coding, the input to the encoder must be exactly recoverable from the output; i.e., the encoder must be invertible.

### 1.6.3 Causality

A system is *causal* if the output at any time depends only on values of the input at the present time and in the past. Such a system is often referred to as being *nonanticipative,* as the system output does not anticipate future values of the input. Consequently, if two inputs to a causal system are identical up to some point in time $t_0$ or $n_0$, the corresponding outputs must also be equal up to this same time. The $RC$ circuit of Figure 1.1 is causal, since the capacitor voltage responds only to the present and past values of the source voltage. Similarly, the motion of an automobile is causal, since it does not anticipate future actions of the driver. The systems described in eqs. (1.92) – (1.94) are also causal, but the systems defined by

$$y[n] = x[n] - x[n+1] \tag{1.102}$$

and

$$y(t) = x(t+1) \tag{1.103}$$

are not. All memoryless systems are causal, since the output responds only to the current value of the input.

Although causal systems are of great importance, they do not by any means constitute the only systems that are of practical significance. For example, causality is not often an essential constraint in applications in which the independent variable is not time, such as in image processing. Furthermore, in processing data that have been recorded previously, as often happens with speech, geophysical, or meteorological signals, to name a few, we are by no means constrained to causal processing. As another example, in many applications, including historical stock market analysis and demographic studies, we may be interested in determining a slowly varying trend in data that also contain high-frequency fluctuations about that trend. In this case, a commonly used approach is to average data over an interval in order to smooth out the fluctuations and keep only the trend. An example of a noncausal averaging system is

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{+M} x[n-k]. \tag{1.104}$$

## Example 1.12

When checking the causality of a system, it is important to look carefully at the input-output relation. To illustrate some of the issues involved in doing this, we will check the causality of two particular systems.

The first system is defined by

$$y[n] = x[-n]. \tag{1.105}$$

Note that the output $y[n_0]$ at a positive time $n_0$ depends only on the value of the input signal $x[-n_0]$ at time $(-n_0)$, which is negative and therefore in the past of $n_0$. We may be tempted to conclude at this point that the given system is causal. However, we should always be careful to check the input-output relation for *all* times. In particular, for $n < 0$, e.g. $n = -4$, we see that $y[-4] = x[4]$, so that the output at this time depends on a future value of the input. Hence, the system is not causal.

It is also important to distinguish carefully the effects of the input from those of any other functions used in the definition of the system. For example, consider the system

$$y(t) = x(t)\cos(t+1). \tag{1.106}$$

In this system, the output at any time $t$ equals the input at that same time multiplied by a number that varies with time. Specifically, we can rewrite eq. (1.106) as

$$y(t) = x(t)g(t),$$

where g(t) is a time-varying function, namely $g(t) = \cos(t + 1)$. Thus, only the current value of the input $x(t)$ influences the current value of the output $y(t)$, and we conclude that this system is causal (and, in fact, memoryless).

### 1.6.4 Stability

*Stability* is another important system property. Informally, a stable system is one in which small inputs lead to responses that do not diverge. For example, consider the pendulum in Figure 1.46(a), in which the input is the applied force $x(t)$ and the output is the angular deviation $y(t)$ from the vertical. In this case, gravity applies a restoring force that tends to return the pendulum to the vertical position, and frictional losses due to drag tend to slow it down. Consequently, if a small force $x(t)$ is applied, the resulting deflection from vertical will also be small. In contrast, for the inverted pendulum in Figure 1.46(b), the effect of gravity is to apply a force that tends to *increase* the deviation from vertical. Thus, a small applied force leads to a large vertical deflection causing the pendulum to topple over, despite any retarding forces due to friction.

The system in Figure 1.46(a) is an example of a stable system, while that in Figure 1.46(b) is unstable. Models for chain reactions or for population growth with unlimited food supplies and no predators are examples of unstable systems, since the system response grows without bound in response to small inputs. Another example of an unstable system is the model for a bank account balance in eq. (1.86), since if an initial deposit is made (i.e., $x[0]$ = a positive amount) and there are no subsequent withdrawals, then that deposit will grow each month without bound, because of the compounding effect of interest payments.

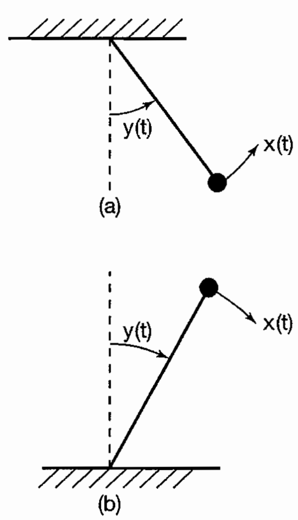



**Figure 1.46** (a) A stable pendulum; (b) an unstable inverted pendulum.

There are also numerous examples of stable systems. Stability of physical systems generally results from the presence of mechanisms that dissipate energy. For example, assuming positive component values in the simple $RC$ circuit of Example 1.8, the resistor dissipates energy and this circuit is a stable system. The system in Example 1.9 is also stable because of the dissipation of energy through friction.

The preceding examples provide us with an intuitive understanding of the concept of stability. More formally, if the input to a stable system is bounded (i.e., if its magnitude does not grow without bound), then the output must also be bounded and therefore cannot diverge. This is the definition of stability that we will use throughout this book. For example, consider applying a constant force $f(t) = F$ to the automobile in Figure 1.2, with the vehicle initially at rest. In this case the velocity of the car will increase, but not without bound, since the retarding frictional force also increases with velocity. In fact, the velocity will continue to increase until the frictional force exactly balances the applied force; so, from eq. (1.84), we see that this terminal velocity value $V$ must satisfy

$$\frac{\rho}{m}V = \frac{1}{m}F, \tag{1.107}$$

i.e.,

$$V = \frac{F}{\rho}. \tag{1.108}$$

As another example, consider the discrete-time system defined by eq. (1.104), and suppose that the input $x[n]$ is bounded in magnitude by some number, say, $B$, for all values of $n$. Then the largest possible magnitude for $y[n]$ is also $B$, because $y[n]$ is the average of a finite set of values of the input. Therefore, $y[n]$ is bounded and the system is stable. On the other hand, consider the accumulator described by eq. (1.92). Unlike the system in eq. (1.104), this system sums *all* of the past values of the input rather than just a finite set of values, and the system is unstable, since the sum can grow continually even if $x[n]$ is bounded. For example, if the input to the accumulator is a unit step $u[n]$, the output will be

$$y[n] = \sum_{k=-\infty}^{n} u[k] = (n+1)u[n].$$

That is, $y[0] = 1$, $y[1] = 2$, $y[2] = 3$, and so on, and $y[n]$ grows without bound.

## Example 1.13

If we suspect that a system is unstable, then a useful strategy to verify this is to look for a *specific* bounded input that leads to an unbounded output. Finding one such example enables us to conclude that the given system is unstable. If such an example does not exist or is difficult to find, we must check for stability by using a method that does not utilize specific examples of input signals. To illustrate this approach, let us check the stability of two systems,

$$S_1: y(t) = tx(t) \tag{1.109}$$

and

$$S_2: y(t) = e^{x(t)}. \tag{1.110}$$

In seeking a specific counterexample in order to disprove stability, we might try simple bounded inputs such as a constant or a unit step. For system $S_1$ in eq. (1.109), a constant input $x(t) = 1$ yields $y(t) = t$, which is unbounded, since no matter what finite constant we pick, $|y(t)|$ will exceed that constant for some $t$. We conclude that system $S_1$ is unstable.

For system $S_2$, which happens to be stable, we would be unable to find a bounded input that results in an unbounded output. So we proceed to verify that all bounded inputs result in bounded outputs. Specifically, let $B$ be an arbitrary positive number, and let $x(t)$ be an arbitrary signal bounded by $B$; that is, we are making no assumption about $x(t)$, except that

$$|x(t)| < B, \tag{1.111}$$

or

$$-B < x(t) < B, \tag{1.112}$$

for all $t$. Using the definition of $S_2$ in eq. (1.110), we then see that if $x(t)$ satisfies eq. (1.111), then $y(t)$ must satisfy

$$e^{-B} < |y(t)| < e^{B}. \tag{1.113}$$

We conclude that if any input to $S_2$ is bounded by an arbitrary positive number $B$, the corresponding output is guaranteed to be bounded by $e^B$. Thus, $S_2$ is stable.

The system properties and concepts that we have introduced so far in this section are of great importance, and we will examine some of these in more detail later in the book. There remain, however, two additional properties—time invariance and linearity—that play a particularly central role in the subsequent chapters of the book, and in the remainder of this section we introduce and provide initial discussions of these two very important concepts.

### 1.6.5 Time Invariance

Conceptually, a system is time invariant if the behavior and characteristics of the system are fixed over time. For example, the *RC* circuit of Figure 1.1 is time invariant if the resistance and capacitance values $R$ and $C$ are constant over time: We would expect to get the same results from an experiment with this circuit today as we would if we ran the identical experiment tomorrow. On the other hand, if the values of $R$ and $C$ are changed or fluctuate over time, then we would expect the results of our experiment to depend on the time at which we run it. Similarly, if the frictional coefficient $b$ and mass $m$ of the automobile in Figure 1.2 are constant, we would expect the vehicle to respond identically independently of when we drive it. On the other hand, if we load the auto's trunk with heavy suitcases one day, thus increasing $m$, we would expect the car to behave differently than at other times when it is not so heavily loaded.

The property of time invariance can be described very simply in terms of the signals and systems language that we have introduced. Specifically, a system is time invariant if a time shift in the input signal results in an identical time shift in the output signal. That is, if $y[n]$ is the output of a discrete-time, time-invariant system when $x[n]$ is the input, then $y[n-n_0]$ is the output when $x[n-n_0]$ is applied. In continuous time with $y(t)$ the output corresponding to the input $x(t)$, a time-invariant system will have $y(t-t_0)$ as the output when $x(t-t_0)$ is the input.

To see how to determine whether a system is time invariant or not, and to gain some insight into this property, consider the following examples:

## Example 1.14

Consider the continuous-time system defined by

$$y(t) = \sin[x(t)]. \tag{1.114}$$

To check that this system is time invariant, we must determine whether the time-invariance property holds for *any* input and *any* time shift $t_0$. Thus, let $x_1(t)$ be an arbitrary input to this system, and let

$$y_1(t) = \sin[x_1(t)] \tag{1.115}$$

be the corresponding output. Then consider a second input obtained by shifting $x_1(t)$ in time:

$$x_2(t) = x_1(t-t_0). \tag{1.116}$$

The output corresponding to this input is

$$y_2(t) = \sin[x_2(t)] = \sin[x_1(t-t_0)]. \tag{1.117}$$

Similarly, from eq. (1.115),

$$y_1(t-t_0) = \sin[x_1(t-t_0)]. \tag{1.118}$$

Comparing eqs. (1.117) and (1.118), we see that $y_2(t) = y_1(t-t_0)$, and therefore, this system is time invariant.

## Example 1.15

As a second example, consider the discrete-time system

$$y[n] = nx[n]. \tag{1.119}$$

This is a time-varying system, a fact that can be verified using the same formal procedure as that used in the preceding example (see Problem 1.28). However, when a system is suspected of being time varying, an approach to showing this that is often very useful is to seek a counterexample—i.e., to use our intuition to find an input signal for which the condition of time invariance is violated. In particular, the system in this example represents a system with a time-varying gain. For example, if we know that the current input value is 1, we cannot determine the current output value without knowing the current time.

Consequently, consider the input signal $x_1[n] = \delta[n]$, which yields an output $y_1[n]$ that is identically 0 (since $n\delta[n] = 0$). However, the input $x_2[n] = \delta[n-1]$ yields the output $y_2[n] = n\delta[n-1] = \delta[n-1]$. Thus, while $x_2[n]$ is a shifted version of $x_1[n]$, $y_2[n]$ is *not* a shifted version of $y_1[n]$.

While the system in the preceding example has a time-varying gain and as a result is a time-varying system, the system in eq. (1.97) has a constant gain and, in fact, is time invariant. Other examples of time-invariant systems are given by eqs. (1.91)–(1.104). The following example illustrates a time-varying system.

## Example 1.16

Consider the system

$$y(t) = x(2t). \tag{1.120}$$

This system represents a time scaling. That is, $y(t)$ is a time-compressed (by a factor of 2) version of $x(t)$. Intuitively, then, any time shift in the input will also be compressed by a factor of 2, and it is for this reason that the system is not time invariant. To demonstrate this by counterexample, consider the input $x_1(t)$ shown in Figure 1.47(a) and the resulting output $y_1(t)$ depicted in Figure 1.47(b). If we then shift the input by 2—i.e., consider $x_2(t) = x_1(t-2)$, as shown in Figure 1.47(c)—we obtain the resulting output

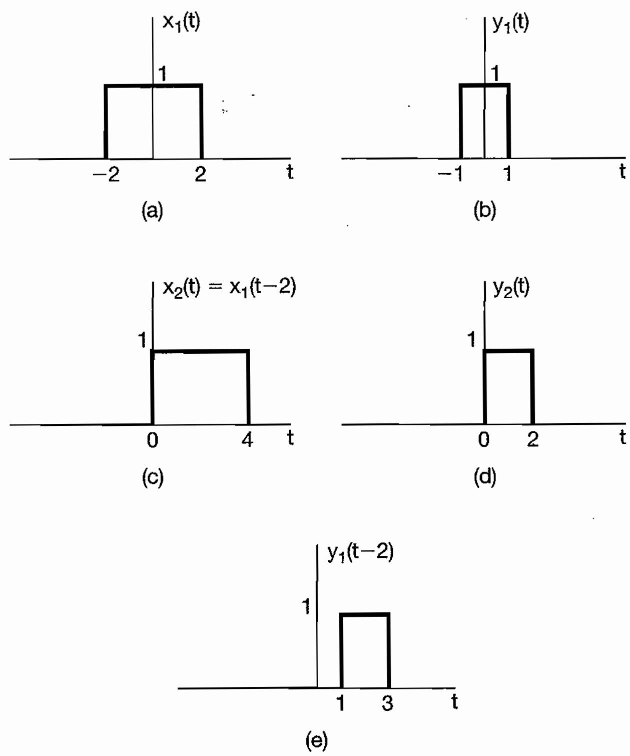



**Figure 1.47** (a) The input $x_1(t)$ to the system in Example 1.16; (b) the output $y_1(t)$ corresponding to $x_1(t)$; (c) the shifted input $x_2(t) = x_1(t-2)$; (d) the output $y_2(t)$ corresponding to $x_2(t)$; (e) the shifted signal $y_1(t-2)$. Note that $y_2(t) \neq y_1(t-2)$, showing that the system is not time invariant.

$y_2(t) = x_2(2t)$ shown in Figure 1.47(d). Comparing Figures 1.47(d) and (e), we see that $y_2(t) \neq y_1(t-2)$, so that the system is not time invariant. (In fact, $y_2(t) = y_1(t-1)$, so that the output time shift is only half as big as it should be for time invariance, due to the time compression imparted by the system.)

### 1.6.6 Linearity

A *linear system*, in continuous time or discrete time, is a system that possesses the important property of superposition: If an input consists of the weighted sum of several signals, then the output is the superposition—that is, the weighted sum—of the responses of the system to each of those signals. More precisely, let $y_1(t)$ be the response of a continuous-time system to an input $x_1(t)$, and let $y_2(t)$ be the output corresponding to the input $x_2(t)$. Then the system is linear if:

1. The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
2. The response to $ax_1(t)$ is $ay_1(t)$, where $a$ is any complex constant.

The first of these two properties is known as the *additivity* property; the second is known as the *scaling* or *homogeneity* property. Although we have written this description using continuous-time signals, the same definition holds in discrete time. The systems specified by eqs. (1.91)–(1.100), (1.102)–(1.104), and (1.119) are linear, while those defined by eqs. (1.101) and (1.114) are nonlinear. Note that a system can be linear without being time invariant, as in eq. (1.119), and it can be time invariant without being linear, as in eqs. (1.101) and (1.114).

The two properties defining a linear system can be combined into a single statement:

$$\text{continuous time: } ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t), \tag{1.121}$$

$$\text{discrete time: } ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]. \tag{1.122}$$

Here, $a$ and $b$ are any complex constants. Furthermore, it is straightforward to show from the definition of linearity that if $x_k[n]$, $k = 1, 2, 3, \ldots$, are a set of inputs to a discrete-time linear system with corresponding outputs $y_k[n]$, $k = 1, 2, 3, \ldots$, then the response to a linear combination of these inputs given by

$$x[n] = \sum_k a_k x_k[n] = a_1x_1[n] + a_2x_2[n] + a_3x_3[n] + \ldots \tag{1.123}$$

is

$$y[n] = \sum_k a_k y_k[n] = a_1y_1[n] + a_2y_2[n] + a_3y_3[n] + \ldots. \tag{1.124}$$

This very important fact is known as the *superposition property*, which holds for linear systems in both continuous and discrete time.

A direct consequence of the superposition property is that, for linear systems, an input which is zero for all time results in an output which is zero for all time. For example, if $x[n] \rightarrow y[n]$, then the homogeneity property tells us that

$$0 = 0 \cdot x[n] \rightarrow 0 \cdot y[n] = 0. \tag{1.125}$$

In the following examples we illustrate how the linearity of a given system can be checked by directly applying the definition of linearity.

### Example 1.17

Consider a system $S$ whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = tx(t)$$

To determine whether or not $S$ is linear, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$x_1(t) \rightarrow y_1(t) = tx_1(t)$$
$$x_2(t) \rightarrow y_2(t) = tx_2(t)$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where $a$ and $b$ are arbitrary scalars. If $x_3(t)$ is the input to $S$, then the corresponding output may be expressed as

$$\begin{aligned} y_3(t) &= tx_3(t) \\ &= t(ax_1(t) + bx_2(t)) \\ &= atx_1(t) + btx_2(t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

We conclude that the system $S$ is linear.

### Example 1.18

Let us apply the linearity-checking procedure of the previous example to another system $S$ whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = x^2(t)$$

Defining $x_1(t)$, $x_2(t)$, and $x_3(t)$ as in the previous example, we have

$$x_1(t) \rightarrow y_1(t) = x_1^2(t)$$
$$x_2(t) \rightarrow y_2(t) = x_2^2(t)$$

and

$$\begin{aligned} x_3(t) \rightarrow y_3(t) &= x_3^2(t) \\ &= (ax_1(t) + bx_2(t))^2 \\ &= a^2x_1^2(t) + b^2x_2^2(t) + 2abx_1(t)x_2(t) \\ &= a^2y_1(t) + b^2y_2(t) + 2abx_1(t)x_2(t) \end{aligned}$$

Clearly, we can specify $x_1(t)$, $x_2(t)$, $a$, and $b$ such that $y_3(t)$ is not the same as $ay_1(t) + by_2(t)$. For example, if $x_1(t) = 1$, $x_2(t) = 0$, $a = 2$, and $b = 0$, then $y_3(t) = (2x_1(t))^2 = 4$, but $2y_1(t) = 2(x_1(t))^2 = 2$. We conclude that the system $S$ is not linear.

### Example 1.19

In checking the linearity of a system, it is important to remember that the system must satisfy both the additivity and homogeneity properties and that the signals, as well as any scaling constants, are allowed to be complex. To emphasize the importance of these points, consider the system specified by

$$y[n] = \mathcal{R}e\{x[n]\}. \tag{1.126}$$

As shown in Problem 1.29, this system is additive; however, it does not satisfy the homogeneity property, as we now demonstrate. Let

$$x_1[n] = r[n] + js[n] \tag{1.127}$$

be an arbitrary complex input with real and imaginary parts $r[n]$ and $s[n]$, respectively, so that the corresponding output is

$$y_1[n] = r[n]. \tag{1.128}$$

Now, consider scaling $x_1[n]$ by a complex number, for example, $a = j$; i.e., consider the input

$$\begin{aligned} x_2[n] &= jx_1[n] = j(r[n] + js[n]) \\ &= -s[n] + jr[n]. \end{aligned} \tag{1.129}$$

The output corresponding to $x_2[n]$ is

$$y_2[n] = \mathcal{R}e\{x_2[n]\} = -s[n], \tag{1.130}$$

which is not equal to the scaled version of $y_1[n]$,

$$ay_1[n] = jr[n]. \tag{1.131}$$

We conclude that the system violates the homogeneity property and hence is not linear.

## Example 1.20

Consider the system

$$y[n] = 2x[n] + 3. \tag{1.132}$$

This system is not linear, as can be verified in several ways. For example, the system violates the additivity property: If $x_1[n] = 2$ and $x_2[n] = 3$, then

$$x_1[n] \to y_1[n] = 2x_1[n] + 3 = 7, \tag{1.133}$$

$$x_2[n] \to y_2[n] = 2x_2[n] + 3 = 9. \tag{1.134}$$

However, the response to $x_3[n] = x_1[n] + x_2[n]$ is

$$y_3[n] = 2[x_1[n] + x_2[n]] + 3 = 13, \tag{1.135}$$

which does not equal $y_1[n] + y_2[n] = 16$. Alternatively, since $y[n] = 3$ if $x[n] = 0$, we see that the system violates the "zero-in/zero-out" property of linear systems given in eq. (1.125).

It may seem surprising that the system in the above example is nonlinear, since eq. (1.132) is a linear equation. On the other hand, as depicted in Figure 1.48, the output of this system can be represented as the sum of the output of a linear system and another signal equal to the *zero-input response* of the system. For the system in eq. (1.132), the linear system is

$$x[n] \to 2x[n],$$

and the zero-input response is

$$y_0[n] = 3.$$

**1.10.** Determine the fundamental period of the signal $x(t) = 2\cos(10t + 1) - \sin(4t - 1)$.

**1.11.** Determine the fundamental period of the signal $x[n] = 1 + e^{j4\pi n/7} - e^{j2\pi n/5}$.

**1.12.** Consider the discrete-time signal

$$x[n] = 1 - \sum_{k=3}^{\infty} \delta[n - 1 - k].$$

Determine the values of the integers $M$ and $n_0$ so that $x[n]$ may be expressed as

$$x[n] = u[Mn - n_0].$$

**1.13.** Consider the continuous-time signal

$$x(t) = \delta(t + 2) - \delta(t - 2).$$

Calculate the value of $E_\infty$ for the signal

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

**1.14.** Consider a periodic signal

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ -2, & 1 < t < 2 \end{cases}$$

with period $T = 2$. The derivative of this signal is related to the "impulse train"

$$g(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k)$$

with period $T = 2$. It can be shown that

$$\frac{dx(t)}{dt} = A_1 g(t - t_1) + A_2 g(t - t_2).$$

Determine the values of $A_1$, $t_1$, $A_2$, and $t_2$.

**1.15.** Consider a system $S$ with input $x[n]$ and output $y[n]$. This system is obtained through a series interconnection of a system $S_1$ followed by a system $S_2$. The input-output relationships for $S_1$ and $S_2$ are

$$S_1: \quad y_1[n] = 2x_1[n] + 4x_1[n - 1],$$
$$S_2: \quad y_2[n] = x_2[n - 2] + \frac{1}{2}x_2[n - 3],$$

where $x_1[n]$ and $x_2[n]$ denote input signals.

**(a)** Determine the input-output relationship for system $S$.

**(b)** Does the input-output relationship of system $S$ change if the order in which $S_1$ and $S_2$ are connected in series is reversed (i.e., if $S_2$ follows $S_1$)?

**1.16.** Consider a discrete-time system with input $x[n]$ and output $y[n]$. The input-output relationship for this system is

$$y[n] = x[n]x[n - 2].$$

**(a)** Is the system memoryless?
**(b)** Determine the output of the system when the input is $A\delta[n]$, where $A$ is any real or complex number.
**(c)** Is the system invertible?

**1.17.** Consider a continuous-time system with input $x(t)$ and output $y(t)$ related by

$$y(t) = x(\sin(t)).$$

**(a)** Is this system causal?
**(b)** Is this system linear?

**1.18.** Consider a discrete-time system with input $x[n]$ and output $y[n]$ related by

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k],$$

where $n_0$ is a finite positive integer.
**(a)** Is this system linear?
**(a)** Is this system time-invariant?
**(c)** If $x[n]$ is known to be bounded by a finite integer $B$ (i.e., $|x[n]| < B$ for all $n$), it can be shown that $y[n]$ is bounded by a finite number $C$. We conclude that the given system is stable. Express $C$ in terms of $B$ and $n_0$.

**1.19.** For each of the following input-output relationships, determine whether the corresponding system is linear, time invariant or both.
**(a)** $y(t) = t^2 x(t-1)$ **(b)** $y[n] = x^2[n-2]$
**(c)** $y[n] = x[n+1] - x[n-1]$ **(d)** $y[n] = \mathcal{O}d\{x(t)\}$

**1.20.** A continuous-time linear system $S$ with input $x(t)$ and output $y(t)$ yields the following input-output pairs:

$$x(t) = e^{j2t} \xrightarrow{S} y(t) = e^{j3t},$$
$$x(t) = e^{-j2t} \xrightarrow{S} y(t) = e^{-j3t}.$$

**(a)** If $x_1(t) = \cos(2t)$, determine the corresponding output $y_1(t)$ for system $S$.
**(b)** If $x_2(t) = \cos(2(t - \frac{1}{2}))$, determine the corresponding output $y_2(t)$ for system $S$.

## BASIC PROBLEMS

**1.21.** A continuous-time signal $x(t)$ is shown in Figure P1.21. Sketch and label carefully each of the following signals:
**(a)** $x(t-1)$ **(b)** $x(2-t)$ **(c)** $x(2t+1)$
**(d)** $x(4-\frac{t}{2})$ **(e)** $[x(t) + x(-t)]u(t)$ **(f)** $x(t)[\delta(t+\frac{3}{2}) - \delta(t-\frac{3}{2})]$

**1.22.** A discrete-time signal is shown in Figure P1.22. Sketch and label carefully each of the following signals:
**(a)** $x[n-4]$ **(b)** $x[3-n]$ **(c)** $x[3n]$
**(d)** $x[3n+1]$ **(e)** $x[n]u[3-n]$ **(f)** $x[n-2]\delta[n-2]$
**(g)** $\frac{1}{2}x[n] + \frac{1}{2}(-1)^n x[n]$ **(h)** $x[(n-1)^2]$

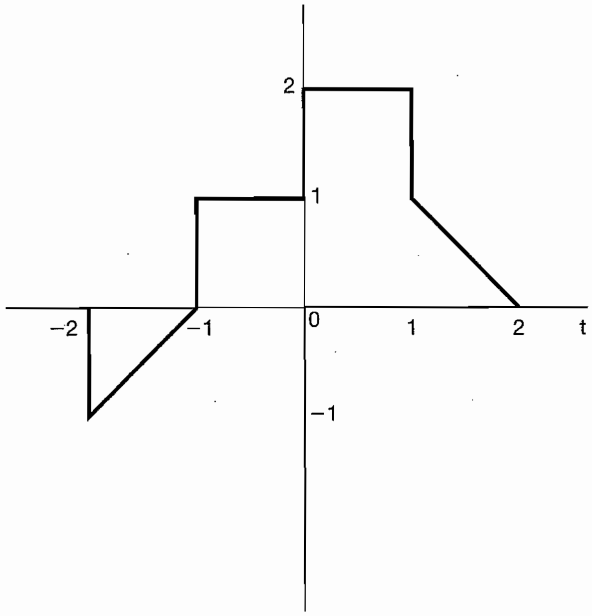



Figure P1.21

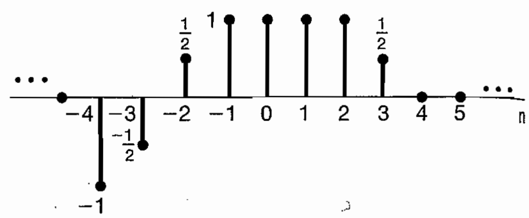



Figure P1.22

**1.23.** Determine and sketch the even and odd parts of the signals depicted in Figure P1.23. Label your sketches carefully.

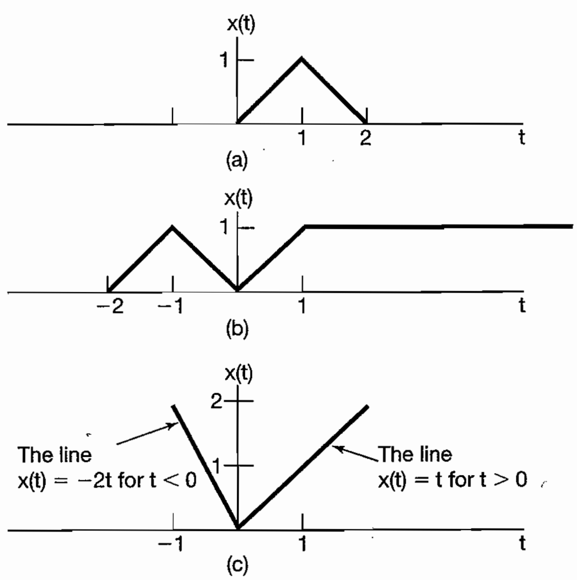



Figure P1.23

**1.24.** Determine and sketch the even and odd parts of the signals depicted in Figure P1.24. Label your sketches carefully.

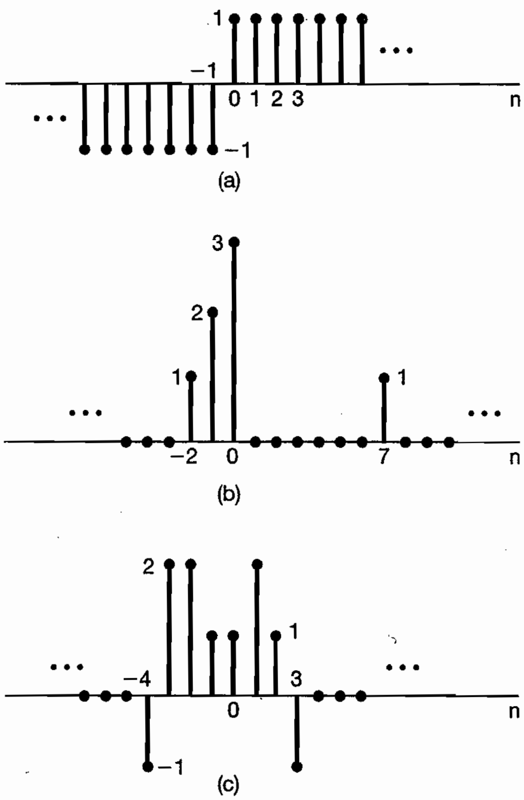



**Figure P1.24**

**1.25.** Determine whether or not each of the following continuous-time signals is periodic. If the signal is periodic, determine its fundamental period.

**(a)** $x(t) = 3\cos(4t + \frac{\pi}{3})$ **(b)** $x(t) = e^{j(\pi t - 1)}$

**(c)** $x(t) = [\cos(2t - \frac{\pi}{3})]^2$ **(d)** $x(t) = \mathcal{E}v\{\cos(4\pi t)u(t)\}$

**(e)** $x(t) = \mathcal{E}v\{\sin(4\pi t)u(t)\}$ **(f)** $x(t) = \sum_{n=-\infty}^{\infty} e^{-(2t-n)} u(2t - n)$

**1.26.** Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period.

**(a)** $x[n] = \sin(\frac{6\pi}{7}n + 1)$ **(b)** $x[n] = \cos(\frac{n}{8} - \pi)$ **(c)** $x[n] = \cos(\frac{\pi}{8}n^2)$

**(d)** $x[n] = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n)$ **(e)** $x[n] = 2\cos(\frac{\pi}{4}n) + \sin(\frac{\pi}{8}n) - 2\cos(\frac{\pi}{2}n + \frac{\pi}{6})$

**1.27.** In this chapter, we introduced a number of general properties of systems. In particular, a system may or may not be

**(1)** Memoryless
**(2)** Time invariant
**(3)** Linear
**(4)** Causal
**(5)** Stable

Determine which of these properties hold and which do not hold for each of the following continuous-time systems. Justify your answers. In each example, $y(t)$ denotes the system output and $x(t)$ is the system input.

(a) $y(t) = x(t-2) + x(2-t)$ (b) $y(t) = [\cos(3t)]x(t)$

(c) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$ (d) $y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-2), & t \geq 0 \end{cases}$

(e) $y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-2), & x(t) \geq 0 \end{cases}$ (f) $y(t) = x(t/3)$

(g) $y(t) = \frac{dx(t)}{dt}$

**1.28.** Determine which of the properties listed in Problem 1.27 hold and which do not hold for each of the following discrete-time systems. Justify your answers. In each example, $y[n]$ denotes the system output and $x[n]$ is the system input.

(a) $y[n] = x[-n]$ (b) $y[n] = x[n-2] - 2x[n-8]$

(c) $y[n] = nx[n]$ (d) $y[n] = \mathcal{E}v\{x[n-1]\}$

(e) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n+1], & n \leq -1 \end{cases}$ (f) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$

(g) $y[n] = x[4n+1]$

**1.29.** (a) Show that the discrete-time system whose input $x[n]$ and output $y[n]$ are related by $y[n] = \mathcal{R}e\{x[n]\}$ is additive. Does this system remain additive if its input-output relationship is changed to $y[n] = \mathcal{R}e\{e^{j\pi n/4}x[n]\}$? (Do not assume that $x[n]$ is real in this problem.)

(b) In the text, we discussed the fact that the property of linearity for a system is equivalent to the system possessing both the additivity property and homogeneity property. Determine whether each of the systems defined below is additive and/or homogeneous. Justify your answers by providing a proof for each property if it holds or a counterexample if it does not.

(i) $y(t) = \frac{1}{x(t)}\left[\frac{dx(t)}{dt}\right]^2$ (ii) $y[n] = \begin{cases} \frac{x[n]x[n-2]}{x[n-1]}, & x[n-1] \neq 0 \\ 0, & x[n-1] = 0 \end{cases}$

**1.30.** Determine if each of the following systems is invertible. If it is, construct the inverse system. If it is not, find two input signals to the system that have the same output.

(a) $y(t) = x(t-4)$ (b) $y(t) = \cos[x(t)]$

(c) $y[n] = nx[n]$ (d) $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

(e) $y[n] = \begin{cases} x[n-1], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$ (f) $y[n] = x[n]x[n-1]$

(g) $y[n] = x[1-n]$ (h) $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau)d\tau$

(i) $y[n] = \sum_{k=-\infty}^{n} (\frac{1}{2})^{n-k}x[k]$ (j) $y(t) = \frac{dx(t)}{dt}$

(k) $y[n] = \begin{cases} x[n+1], & n \geq 0 \\ x[n], & n \leq -1 \end{cases}$ (l) $y(t) = x(2t)$

(m) $y[n] = x[2n]$ (n) $y[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

**1.31.** In this problem, we illustrate one of the most important consequences of the properties of linearity and time invariance. Specifically, once we know the response of a linear system or a linear time-invariant (LTI) system to a single input or the responses to several inputs, we can directly compute the responses to many other