(a) Is the system memoryless?
(b) Determine the output of the system when the input is $A\delta[n]$, where A is any real or complex number.
(c) Is the system invertible?

1.17. Consider a continuous-time system with input $x(t)$ and output $y(t)$ related by

$$y(t) = x(\sin(t)).$$

(a) Is this system causal?
(b) Is this system linear?

1.18. Consider a discrete-time system with input $x[n]$ and output $y[n]$ related by

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k],$$

where n_0 is a finite positive integer.
(a) Is this system linear?
(a) Is this system time-invariant?
(c) If $x[n]$ is known to be bounded by a finite integer B (i.e., $|x[n]| < B$ for all n), it can be shown that $y[n]$ is bounded by a finite number C. We conclude that the given system is stable. Express C in terms of B and n_0.

1.19. For each of the following input-output relationships, determine whether the corresponding system is linear, time invariant or both.
(a) $y(t) = t^2 x(t-1)$ **(b)** $y[n] = x^2[n-2]$
(c) $y[n] = x[n+1] - x[n-1]$ **(d)** $y[n] = \mathcal{O}d\{x(t)\}$

1.20. A continuous-time linear system S with input $x(t)$ and output $y(t)$ yields the following input-output pairs:

$$x(t) = e^{j2t} \xrightarrow{S} y(t) = e^{j3t},$$

$$x(t) = e^{-j2t} \xrightarrow{S} y(t) = e^{-j3t}.$$

(a) If $x_1(t) = \cos(2t)$, determine the corresponding output $y_1(t)$ for system S.
(b) If $x_2(t) = \cos(2(t - \frac{1}{2}))$, determine the corresponding output $y_2(t)$ for system S.

BASIC PROBLEMS

1.21. A continuous-time signal $x(t)$ is shown in Figure P1.21. Sketch and label carefully each of the following signals:
(a) $x(t-1)$ **(b)** $x(2-t)$ **(c)** $x(2t+1)$
(d) $x(4 - \frac{t}{2})$ **(e)** $[x(t) + x(-t)]u(t)$ **(f)** $x(t)[\delta(t + \frac{3}{2}) - \delta(t - \frac{3}{2})]$

1.22. A discrete-time signal is shown in Figure P1.22. Sketch and label carefully each of the following signals:
(a) $x[n-4]$ **(b)** $x[3-n]$ **(c)** $x[3n]$
(d) $x[3n+1]$ **(e)** $x[n]u[3-n]$ **(f)** $x[n-2]\delta[n-2]$
(g) $\frac{1}{2}x[n] + \frac{1}{2}(-1)^n x[n]$ **(h)** $x[(n-1)^2]$

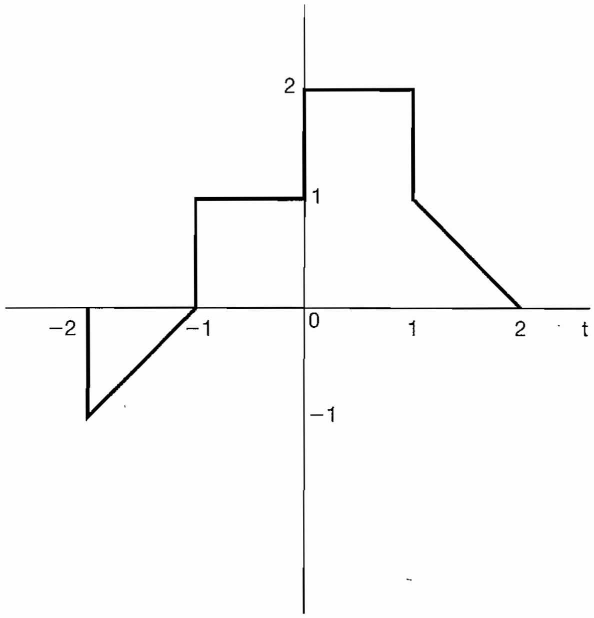

Figure P1.21

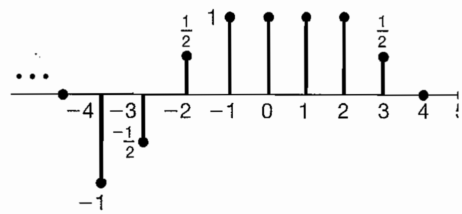

Figure P1.22

1.23. Determine and sketch the even and odd parts of the signals depicted in Figure P1
Label your sketches carefully.

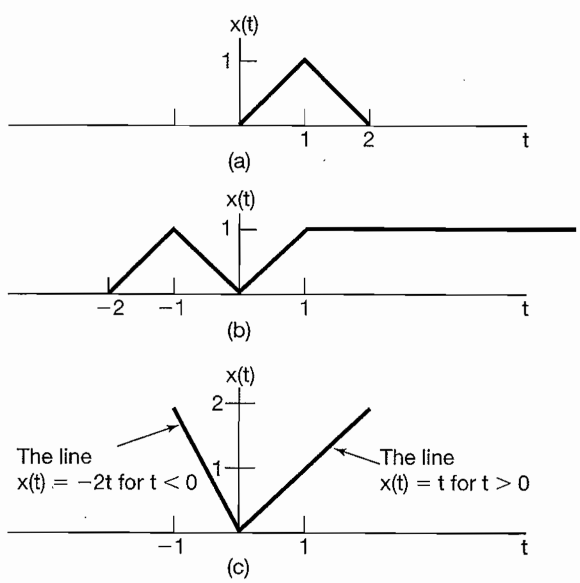

Figure P1.23

1.24. Determine and sketch the even and odd parts of the signals depicted in Figure P1
Label your sketches carefully.

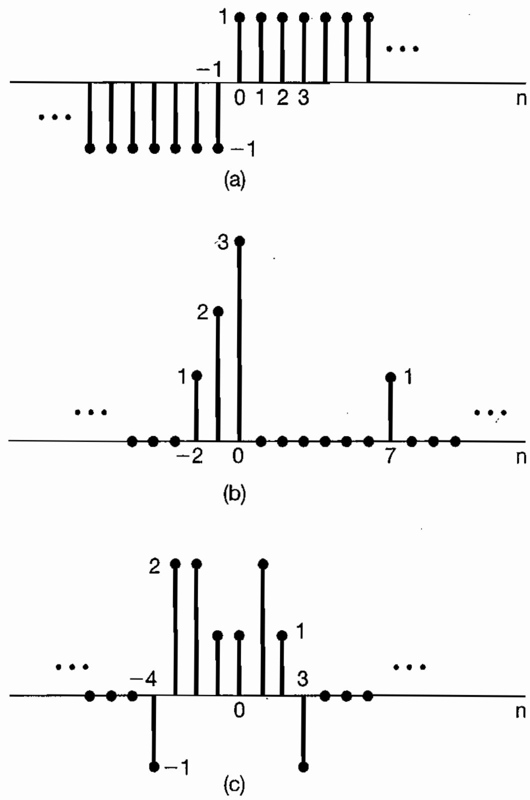

Figure P1.24

1.25. Determine whether or not each of the following continuous-time signals is periodic. If the signal is periodic, determine its fundamental period.

(a) $x(t) = 3\cos(4t + \frac{\pi}{3})$ **(b)** $x(t) = e^{j(\pi t - 1)}$

(c) $x(t) = [\cos(2t - \frac{\pi}{3})]^2$ **(d)** $x(t) = \mathcal{E}v\{\cos(4\pi t)u(t)\}$

(e) $x(t) = \mathcal{E}v\{\sin(4\pi t)u(t)\}$ **(f)** $x(t) = \sum_{n=-\infty}^{\infty} e^{-(2t-n)}u(2t-n)$

1.26. Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period.

(a) $x[n] = \sin(\frac{6\pi}{7}n + 1)$ **(b)** $x[n] = \cos(\frac{n}{8} - \pi)$ **(c)** $x[n] = \cos(\frac{\pi}{8}n^2)$

(d) $x[n] = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n)$ **(e)** $x[n] = 2\cos(\frac{\pi}{4}n) + \sin(\frac{\pi}{8}n) - 2\cos(\frac{\pi}{2}n + \frac{\pi}{6})$

1.27. In this chapter, we introduced a number of general properties of systems. In particular, a system may or may not be

(1) Memoryless
(2) Time invariant
(3) Linear
(4) Causal
(5) Stable

Determine which of these properties hold and which do not hold for each of the following continuous-time systems. Justify your answers. In each example, $y(t)$ denotes the system output and $x(t)$ is the system input.

(a) $y(t) = x(t-2) + x(2-t)$

(b) $y(t) = [\cos(3t)]x(t)$

(c) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$

(d) $y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-2), & t \geq 0 \end{cases}$

(e) $y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-2), & x(t) \geq 0 \end{cases}$

(f) $y(t) = x(t/3)$

(g) $y(t) = \frac{dx(t)}{dt}$

1.28. Determine which of the properties listed in Problem 1.27 hold and which do not hold for each of the following discrete-time systems. Justify your answers. In each example, $y[n]$ denotes the system output and $x[n]$ is the system input.

(a) $y[n] = x[-n]$

(b) $y[n] = x[n-2] - 2x[n-8]$

(c) $y[n] = nx[n]$

(d) $y[n] = \mathcal{E}v\{x[n-1]\}$

(e) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n+1], & n \leq -1 \end{cases}$

(f) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$

(g) $y[n] = x[4n+1]$

1.29. (a) Show that the discrete-time system whose input $x[n]$ and output $y[n]$ are related by $y[n] = \mathcal{R}e\{x[n]\}$ is additive. Does this system remain additive if its input-output relationship is changed to $y[n] = \mathcal{R}e\{e^{j\pi n/4}x[n]\}$? (Do not assume that $x[n]$ is real in this problem.)

(b) In the text, we discussed the fact that the property of linearity for a system is equivalent to the system possessing both the additivity property and homogeneity property. Determine whether each of the systems defined below is additive and/or homogeneous. Justify your answers by providing a proof for each property if it holds or a counterexample if it does not.

(i) $y(t) = \frac{1}{x(t)}\left[\frac{dx(t)}{dt}\right]^2$ (ii) $y[n] = \begin{cases} \frac{x[n]x[n-2]}{x[n-1]}, & x[n-1] \neq 0 \\ 0, & x[n-1] = 0 \end{cases}$

1.30. Determine if each of the following systems is invertible. If it is, construct the inverse system. If it is not, find two input signals to the system that have the same output.

(a) $y(t) = x(t-4)$

(b) $y(t) = \cos[x(t)]$

(c) $y[n] = nx[n]$

(d) $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

(e) $y[n] = \begin{cases} x[n-1], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$

(f) $y[n] = x[n]x[n-1]$

(g) $y[n] = x[1-n]$

(h) $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau)d\tau$

(i) $y[n] = \sum_{k=-\infty}^{n} (\frac{1}{2})^{n-k}x[k]$

(j) $y(t) = \frac{dx(t)}{dt}$

(k) $y[n] = \begin{cases} x[n+1], & n \geq 0 \\ x[n], & n \leq -1 \end{cases}$

(l) $y(t) = x(2t)$

(m) $y[n] = x[2n]$

(n) $y[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

1.31. In this problem, we illustrate one of the most important consequences of the properties of linearity and time invariance. Specifically, once we know the response of a linear system or a linear time-invariant (LTI) system to a single input or the responses to several inputs, we can directly compute the responses to many other input signals. Much of the remainder of this book deals with a thorough exploitation of this fact in order to develop results and techniques for analyzing and synthesizing LTI systems.

(a) Consider an LTI system whose response to the signal $x_1(t)$ in Figure P1.31(a) is the signal $y_1(t)$ illustrated in Figure P1.31(b). Determine and sketch carefully the response of the system to the input $x_2(t)$ depicted in Figure P1.31(c).

(b) Determine and sketch the response of the system considered in part (a) to the input $x_3(t)$ shown in Figure P1.31(d).

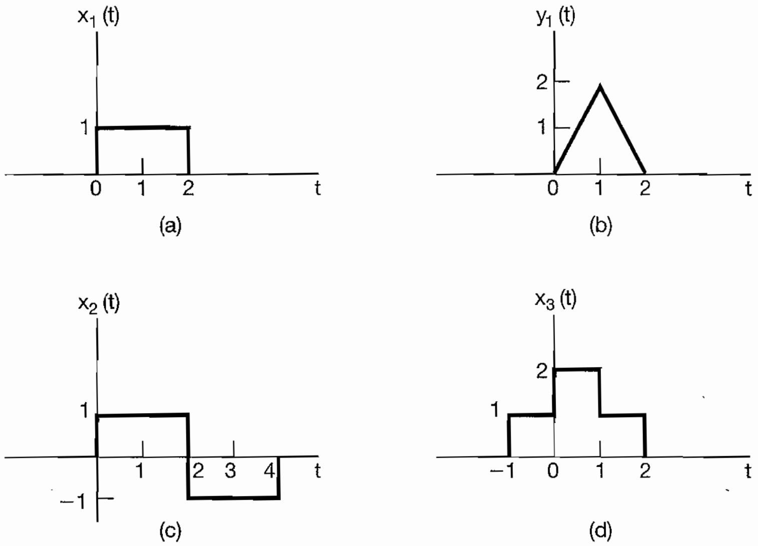

Figure P1.31

ADVANCED PROBLEMS

1.32. Let $x(t)$ be a continuous-time signal, and let

$$y_1(t) = x(2t) \text{ and } y_2(t) = x(t/2).$$

The signal $y_1(t)$ represents a speeded up version of $x(t)$ in the sense that the duration of the signal is cut in half. Similarly, $y_2(t)$ represents a slowed down version of $x(t)$ in the sense that the duration of the signal is doubled. Consider the following statements:

(1) If $x(t)$ is periodic, then $y_1(t)$ is periodic.
(2) If $y_1(t)$ is periodic, then $x(t)$ is periodic.
(3) If $x(t)$ is periodic, then $y_2(t)$ is periodic.
(4) If $y_2(t)$ is periodic, then $x(t)$ is periodic.

For each of these statements, determine whether it is true, and if so, determine the relationship between the fundamental periods of the two signals considered in the statement. If the statement is not true, produce a counterexample to it.

1.33. Let $x[n]$ be a discrete-time signal, and let

$$y_1[n] = x[2n] \text{ and } y_2[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}.$$

The signals $y_1[n]$ and $y_2[n]$ respectively represent in some sense the speeded up and slowed down versions of $x[n]$. However, it should be noted that the discrete-time notions of speeded up and slowed down have subtle differences with respect to their continuous-time counterparts. Consider the following statements:

(1) If $x[n]$ is periodic, then $y_1[n]$ is periodic.
(2) If $y_1[n]$ is periodic, then $x[n]$ is periodic.
(3) If $x[n]$ is periodic, then $y_2[n]$ is periodic.
(4) If $y_2[n]$ is periodic, then $x[n]$ is periodic.

For each of these statements, determine whether it is true, and if so, determine the relationship between the fundamental periods of the two signals considered in the statement. If the statement is not true, produce a counterexample to it.

1.34. In this problem, we explore several of the properties of even and odd signals.

(a) Show that if $x[n]$ is an odd signal, then

$$\sum_{n=-\infty}^{+\infty} x[n] = 0.$$

(b) Show that if $x_1[n]$ is an odd signal and $x_2[n]$ is an even signal, then $x_1[n]x_2[n]$ is an odd signal.

(c) Let $x[n]$ be an arbitrary signal with even and odd parts denoted by

$$x_e[n] = \mathcal{E}v\{x[n]\}$$

and

$$x_o[n] = \mathcal{O}d\{x[n]\}.$$

Show that

$$\sum_{n=-\infty}^{+\infty} x^2[n] = \sum_{n=-\infty}^{+\infty} x_e^2[n] + \sum_{n=-\infty}^{+\infty} x_o^2[n].$$

(d) Although parts (a)–(c) have been stated in terms of discrete-time signals, the analogous properties are also valid in continuous time. To demonstrate this, show that

$$\int_{-\infty}^{+\infty} x^2(t)dt = \int_{-\infty}^{+\infty} x_e^2(t)dt + \int_{-\infty}^{+\infty} x_o^2(t)dt,$$

where $x_e(t)$ and $x_o(t)$ are, respectively, the even and odd parts of $x(t)$.

1.35. Consider the periodic discrete-time exponential time signal

$$x[n] = e^{jm(2\pi/N)n}.$$

Show that the fundamental period of this signal is

$$N_0 = N/\gcd(m, N),$$

where $\gcd(m, N)$ is the *greatest common divisor* of m and N—that is, the largest integer that divides both m and N an integral number of times. For example,

$$\gcd(2, 3) = 1, \gcd(2, 4) = 2, \gcd(8, 12) = 4.$$

Note that $N_0 = N$ if m and N have no factors in common.

1.36. Let $x(t)$ be the continuous-time complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

with fundamental frequency ω_0 and fundamental period $T_0 = 2\pi/\omega_0$. Consider the discrete-time signal obtained by taking equally spaced samples of $x(t)$—that is,

$$x[n] = x(nT) = e^{j\omega_0 nT}.$$

(a) Show that $x[n]$ is periodic if and only if T/T_0 is a rational number—that is, if and only if some multiple of the sampling interval *exactly equals* a multiple of the period of $x(t)$.

(b) Suppose that $x[n]$ is periodic—that is, that

$$\frac{T}{T_0} = \frac{p}{q}, \tag{P1.36–1}$$

where p and q are integers. What are the fundamental period and fundamental frequency of $x[n]$? Express the fundamental frequency as a fraction of $\omega_0 T$.

(c) Again assuming that T/T_0 satisfies eq. (P1.36–1), determine precisely how many periods of $x(t)$ are needed to obtain the samples that form a single period of $x[n]$.

1.37. An important concept in many communications applications is the *correlation* between two signals. In the problems at the end of Chapter 2, we will have more to say about this topic and will provide some indication of how it is used in practice. For now, we content ourselves with a brief introduction to correlation functions and some of their properties.

Let $x(t)$ and $y(t)$ be two signals; then the *correlation function* is defined as

$$\phi_{xy}(t) = \int_{-\infty}^{\infty} x(t+\tau)y(\tau)d\tau.$$

The function $\phi_{xx}(t)$ is usually referred to as the *autocorrelation function* of the signal $x(t)$, while $\phi_{xy}(t)$ is often called a *cross-correlation function.*

(a) What is the relationship between $\phi_{xy}(t)$ and $\phi_{yx}(t)$?

(b) Compute the odd part of $\phi_{xx}(t)$.

(c) Suppose that $y(t) = x(t+T)$. Express $\phi_{xy}(t)$ and $\phi_{yy}(t)$ in terms of $\phi_{xx}(t)$.

1.38. In this problem, we examine a few of the properties of the unit impulse function.

(a) Show that

$$\delta(2t) = \frac{1}{2}\delta(t).$$

Hint: Examine $\delta_\Delta(t)$. (See Figure 1.34.)

(b) In Section 1.4, we defined the continuous-time unit impulse as the limit of the signal $\delta_\Delta(t)$. More precisely, we defined several of the *properties* of $\delta(t)$ by examining the corresponding properties of $\delta_\Delta(t)$. For example, since the signal

$$u_\Delta(t) = \int_{-\infty}^{t} \delta_\Delta(\tau)d\tau$$

1.48. Let z_0 be a complex number with polar coordinates (r_0, θ_0) and Cartesian coordinates (x_0, y_0). Determine expressions for the Cartesian coordinates of the following complex numbers in terms of x_0 and y_0. Plot the points z_0, z_1, z_2, z_3, z_4, and z_5 in the complex plane when $r_0 = 2$ and $\theta_0 = \pi/4$ and when $r_0 = 2$ and $\theta_0 = \pi/2$. Indicate on your plots the real and imaginary parts of each point.

(a) $z_1 = r_0 e^{-j\theta_0}$ **(b)** $z_2 = r_0$ **(c)** $z_3 = r_0 e^{j(\theta_0+\pi)}$
(d) $z_4 = r_0 e^{j(-\theta_0+\pi)}$ **(e)** $z_5 = r_0 e^{j(\theta_0+2\pi)}$

1.49. Express each of the following complex numbers in polar form, and plot them in the complex plane, indicating the magnitude and angle of each number:

(a) $1 + j\sqrt{3}$ **(b)** -5 **(c)** $-5 - 5j$
(d) $3 + 4j$ **(e)** $(1 - j\sqrt{3})^3$ **(f)** $(1 + j)^5$
(g) $(\sqrt{3} + j^3)(1 - j)$ **(h)** $\frac{2-j(6/\sqrt{3})}{2+j(6/\sqrt{3})}$ **(i)** $\frac{1+j\sqrt{3}}{\sqrt{3}+j}$
(j) $j(1 + j)e^{j\pi/6}$ **(k)** $(\sqrt{3} + j)2\sqrt{2}e^{-j\pi/4}$ **(l)** $\frac{e^{j\pi/3}-1}{1+j\sqrt{3}}$

1.50. **(a)** Using Euler's relationship or Figure P1.48, determine expressions for x and y in terms of r and θ.

(b) Determine expressions for r and θ in terms of x and y.

(c) If we are given only r and $\tan\theta$, can we uniquely determine x and y? Explain your answer.

1.51. Using Euler's relation, derive the following relationships:

(a) $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$
(b) $\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$
(c) $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$
(d) $(\sin\theta)(\sin\phi) = \frac{1}{2}\cos(\theta - \phi) - \frac{1}{2}\cos(\theta + \phi)$
(e) $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

1.52. Let z denote a complex variable; that is,

$$z = x + jy = re^{j\theta}.$$

The *complex conjugate* of z is

$$z^* = x - jy = re^{-j\theta}.$$

Derive each of the following relations, where z, z_1, and z_2 are arbitrary complex numbers:

(a) $zz^* = r^2$
(b) $\frac{z}{z^*} = e^{j2\theta}$
(c) $z + z^* = 2\mathcal{R}e\{z\}$
(d) $z - z^* = 2j\mathcal{I}m\{z\}$
(e) $(z_1 + z_2)^* = z_1^* + z_2^*$
(f) $(az_1z_2)^* = az_1^*z_2^*$, where a is any real number
(g) $(\frac{z_1}{z_2})^* = \frac{z_1^*}{z_2^*}$
(h) $\mathcal{R}e\{\frac{z_1}{z_2}\} = \frac{1}{2}[\frac{z_1z_2^* + z_1^*z_2}{z_2z_2^*}]$

1.53. Derive the following relations, where z, z_1, and z_2 are arbitrary complex numbers:

(a) $(e^z)^* = e^{z^*}$
(b) $z_1z_2^* + z_1^*z_2 = 2\mathcal{R}e\{z_1z_2^*\} = 2\mathcal{R}e\{z_1^*z_2\}$

(c) $|z| = |z^*|$
(d) $|z_1 z_2| = |z_1||z_2|$
(e) $\mathcal{R}e\{z\} \leq |z|, \mathcal{I}m\{z\} \leq |z|$
(f) $|z_1 z_2^* + z_1^* z_2| \leq 2|z_1 z_2|$
(g) $(|z_1| - |z_2|)^2 \leq |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$

1.54. The relations considered in this problem are used on many occasions throughout the book.

(a) Prove the validity of the following expression:

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \alpha = 1 \\ \frac{1-\alpha^N}{1-\alpha}, & \text{for any complex number } \alpha \neq 1 \end{cases}.$$

This is often referred to as the *finite sum formula.*

(b) Show that if $|\alpha| < 1$, then

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}.$$

This is often referred to as the *infinite sum formula.*

(c) Show also if $|\alpha| < 1$, then

$$\sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2}.$$

(d) Evaluate

$$\sum_{n=k}^{\infty} \alpha^n,$$

assuming that $|\alpha| < 1$.

1.55. Using the results from Problem 1.54, evaluate each of the following sums and express your answer in Cartesian (rectangular) form:

(a) $\sum_{n=0}^{9} e^{j\pi n/2}$ **(b)** $\sum_{n=-2}^{7} e^{j\pi n/2}$
(c) $\sum_{n=0}^{\infty} (\frac{1}{2})^n e^{j\pi n/2}$ **(d)** $\sum_{n=2}^{\infty} (\frac{1}{2})^n e^{j\pi n/2}$
(e) $\sum_{n=0}^{9} \cos(\frac{\pi}{2} n)$ **(f)** $\sum_{n=0}^{\infty} (\frac{1}{2})^n \cos(\frac{\pi}{2} n)$

1.56. Evaluate each of the following integrals, and express your answer in Cartesian (rectangular) form:

(a) $\int_0^4 e^{j\pi t/2} dt$ **(b)** $\int_0^6 e^{j\pi t/2} dt$
(c) $\int_2^8 e^{j\pi t/2} dt$ **(d)** $\int_0^\infty e^{-(1+j)t} dt$
(e) $\int_0^\infty e^{-t} \cos(t) dt$ **(f)** $\int_0^\infty e^{-2t} \sin(3t) dt$