REVISED

Some Proofs:

Proofs for $x(t)$ real-valued

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

If $x(t)$ is real-valued, then $X(-\omega) = X^*(\omega)$

Proof:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Take conjugate of both sides:

$$X^*(\omega) = \int_{-\infty}^{\infty} x(t) e^{+j\omega t} dt$$

($x(t) = x^*(t)$ since real-valued)

- replace ω by $-\omega$ on both sides:

$$X^*(-\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= X(\omega) \Rightarrow \boxed{X^*(-\omega) = X(\omega)}$$

Thus: for $x(t)$ real-valued:

$$X(-\omega) = X^*(\omega)$$

(take conjugate of both sides of $X^*(-\omega) = X(\omega)$)

Expressing $X(\omega)$ in polar form:

$$X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$$

This says: $X^*(-\omega) = |X(-\omega)| e^{-j\angle X(-\omega)}$

$$= X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$$

Hence: if $x(t)$ real-valued

$$|X(-\omega)| = |X(\omega)|$$ magnitude is even fn. of freq.

$$\angle X(-\omega) = -\angle X(\omega)$$ phase is odd fn. of frequency

Proofs for even and odd parts of $x(t)$

We proved in class:

$$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Thus, if $a = -1$

$$x(-t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(-\omega)$$

Now, if $x(t)$ is real-valued, $X(-\omega) = X^*(\omega)$

THUS, for $x(t)$ real-valued:

$$x(-t) \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(\omega)$$

OKAY, so for even part of $x(t)$:

$$x_e(t) = \frac{1}{2}\left\{x(t) + x(-t)\right\}$$

AND IT follows from above:

$$X_e(\omega) = \frac{1}{2}\left\{X(\omega) + X^*(\omega)\right\}$$

$$= \text{Real}\left\{X(\omega)\right\}$$

$$x_e(t) = \mathcal{E}v\{x(t)\} \stackrel{\mathcal{F}}{\longleftrightarrow} \text{Real}\left\{X(\omega)\right\}$$

~~for~~ odd part of $x(t)$:

$$x_o(t) = \frac{1}{2}\left\{x(t) - x(-t)\right\}$$

$$X_o(\omega) = \frac{1}{2}\left\{X(\omega) - X^*(\omega)\right\}$$

$$= j\,\text{Imag}\left\{X(\omega)\right\}$$

$$x_o(t) = \mathcal{O}d\left\{x(t)\right\} \stackrel{\mathcal{F}}{\longleftrightarrow} j\,\text{Im}\left\{X(\omega)\right\}$$

Summarizing:

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega) = Re\{X(\omega)\} + j\, Im\{X(\omega)\}$$

$$x_e(t) = \mathcal{E}v\{x(t)\} = \frac{1}{2}\{x(t) + x(-t)\} \xleftrightarrow{\mathcal{F}} Re\{X(\omega)\}$$

$$x_o(t) = \mathcal{O}d\{x(t)\} = \frac{1}{2}\{x(t) - x(-t)\} \xleftrightarrow{\mathcal{F}} j\, Im\{X(\omega)\}$$

Since $x(t) = x_e(t) + x_o(t)$, sure enough:

$$X(\omega) = \overbrace{Re\{X(\omega)\}} + \overbrace{j\, Im\{X(\omega)\}}$$

The previous page + the top of this page was for an arbitrary real-valued $x(t)$

Now, consider $x(t)$ to be both real and even:

$$x(-t) = x(t)$$

Take Fourier Transform of both sides:

$$X(-\omega) = X(\omega)$$

plus, we proved for $x(t)$ real-valued $X(-\omega) = X^*(\omega)$

Thus, we have $X^*(\omega) = X(\omega)$ and that can only be true if $X(\omega)$ is real-valued.

Thus, if $x(t)$ is real and even, $X(\omega)$ is real and even

For $x(t)$ real and odd, $x(-t) = -x(t) \Rightarrow X(-\omega) = -X(\omega)$

since $X(-\omega) = X^*(\omega)$, $\Rightarrow X^*(\omega) = -X(\omega) \Rightarrow$ only true if $X(\omega)$ is purely imaginary $\Rightarrow X(\omega)$ is purely imaginary and odd

Some properties following from frequency shift property

- We proved in class:

$$x(t)\,e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega-\omega_0)$$

- Now, since

$$\cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$x(t)\cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}X(\omega-\omega_0) + \frac{1}{2}X(\omega+\omega_0)$$

- Further, since: $\sin(\omega_0 t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$

$$x(t)\sin(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2j}X(\omega-\omega_0) - \frac{1}{2j}X(\omega+\omega_0)$$

All follows from Euler's formulas:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$$

$$\sin\theta = \frac{1}{2j}e^{j\theta} - \frac{1}{2j}e^{-j\theta}$$

Prob. 4.21 (e)

$$x(t) = t\, e^{-2t} \sin(4t)\, u(t) \xleftrightarrow{\mathcal{F}} X(\omega) = ?$$

Rewrite as: $x(t) = t\left[e^{-2t} u(t) \cdot \sin(4t)\right]$

$$= t\, z(t)$$

- First, find $Z(\omega)$,

Since from table, $e^{-2t} u(t) \xleftrightarrow{\mathcal{F}} \dfrac{1}{2 + j\omega}$

THUS:

$$Z(\omega) = \mathcal{F}\left\{ e^{-4t} u(t)\, \sin(4t) \right\}$$

$$= \frac{1}{2j}\, \frac{1}{2 + j(\omega - 4)} - \frac{1}{2j}\, \frac{1}{2 + j(\omega + 4)}$$

- Now, from Table:

$$t\, z(t) \xleftrightarrow{\mathcal{F}} j\, \frac{d}{d\omega}\{Z(\omega)\}$$

THUS:

$$X(\omega) = j\, \frac{d}{d\omega}\{Z(\omega)\}$$

$$= \frac{1}{2}\, \frac{d}{d\omega}\left\{\frac{1}{2 + j(\omega - 4)}\right\} + \frac{1}{2}\, \frac{d}{d\omega}\left\{\frac{1}{2 + j(\omega + 4)}\right\}$$

$$= \frac{1}{2}\, \frac{-j}{(2 + j(\omega - 4))^2} + \frac{1}{2}\, \frac{-j}{(2 + j(\omega + 4))^2}$$

Initial Value Theorems:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

time-domain | frequency domain

$$\Rightarrow x(t)\Big|_{t=0} = x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, \cancelto{1}{e^{j0}}\, d\omega$$

so:

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, d\omega$$

Similarly:

$$X(\omega)\Big|_{\omega=0} = \int_{-\infty}^{\infty} x(t)\, \cancelto{1}{e^{j0}}\, dt$$

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

Duality Property: $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$

then $X(t) \xleftrightarrow{\mathcal{F}} 2\pi\, x(-\omega)$

Parts (a) + (b)

Prob. 4.23

$$x_o(t) = \begin{cases} e^{-t}, & 0 < t < 1 \\ 0, & \text{else} \end{cases}$$

As discussed in class, actually easier to do integral in this case:

$$X_o(\omega) = \int_0^1 e^{-t} e^{-j\omega t}\, dt = \int_0^1 e^{-(1+j\omega)t}\, dt$$

$$= \left. \frac{e^{-(1+j\omega)t}}{-(1+j\omega)} \right|_0^1 = \frac{e^{-(1+j\omega)} - e^0}{-(1+j\omega)}$$

$$= \frac{1 - e^{-1} e^{-j\omega}}{1+j\omega} \cdot \frac{(1-j\omega)}{(1-j\omega)}$$

$$= \frac{(1 - e^{-1}\cos\omega + e^{-1}\sin\omega)(1-j\omega)}{1+\omega^2}$$

$$= \underbrace{\frac{2 - 2e^{-1}\cos\omega - 2e^{-1}\omega\sin\omega}{1+\omega^2}}_{} + j\underbrace{\left[\frac{-2\omega + 2e^{-1}\sin\omega + 2e^{-1}\omega\cos\omega}{1+\omega^2}\right]}_{}$$

$$= \quad \mathrm{Re}\{X_o(\omega)\} \quad \text{even fn. of } \omega \qquad \mathrm{Im}\{X_o(\omega)\} \quad \text{odd fn. of } \omega$$

Recall:

$$x_{even}(t) = \frac{1}{2}\left\{x_o(t) + x_o(-t)\right\} \xleftrightarrow{\mathcal{F}} \mathrm{Re}\{X_o(\omega)\}$$

$$x_{odd}(t) = \frac{1}{2}\left\{x_o(t) - x_o(-t)\right\} \xleftrightarrow{\mathcal{F}} j\,\mathrm{Im}\{X_o(\omega)\}$$