Passing sinewaves thru LTI System

$$x(t) = e^{j\omega_0 t} \;\; \forall t \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega)\end{array}} \longrightarrow y(t) = H(\omega_0)\, e^{j\omega_0 t}$$

- Recall: this is where the defining integral equation for the Fourier Transform came from:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, \underbrace{e^{j\omega_0 (t-\tau)}}_{x(t-\tau)}\, d\tau$$

$$= e^{j\omega_0 t} \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega_0 \tau}\, d\tau$$

$$= H(\omega_0)\, e^{j\omega_0 t}$$

- where: $H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$

- THUS: $e^{j\omega_0 t} * h(t) = H(\omega_0)\, e^{j\omega_0 t}$

$\Rightarrow$ no reason to do convolution

- In frequency domain $\Rightarrow$ same result:

$$y(t) = e^{j\omega_0 t} * h(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = 2\pi\,\delta(\omega-\omega_0)\, H(\omega)$$

$$= H(\omega_0)\, e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} = H(\omega_0)\, 2\pi\,\delta(\omega-\omega_0)$$

$\Rightarrow$ no reason to have to do this either on exam (see next page)

Using Euler's Identities

$$\cos(\omega_0 t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

$$\sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

and invoking linearity, can easily show:

$$\cos(\omega_0 t) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega)\end{matrix}} \longrightarrow |H(\omega_0)| \cos(\omega_0 t + \angle H(\omega_0))$$

$$\sin(\omega_0 t) \longrightarrow \boxed{h(t)} \longrightarrow |H(\omega_0)| \sin(\omega_0 t + \angle H(\omega_0))$$

$$\boxed{\begin{matrix} H(\omega_0) = |H(\omega_0)| e^{j\angle H(\omega_0)} \\ \text{polar form} \end{matrix}}$$

Further, invoking linearity:

$$\underbrace{\sum_{k=-\infty}^{\infty} a_k e^{j\omega_k t}}_{\text{sum of sinewaves}} \longrightarrow \boxed{h(t)} \longrightarrow \sum_{k=-\infty}^{\infty} a_k H(\omega_k) e^{j\omega_k t}$$

- special case: periodic signal where $\omega_k = k\frac{2\pi}{T}$

- **FURTHER:**

$$\sum_{k=-\infty}^{\infty} A_k \cos(\omega_k t + \theta_k) \longrightarrow \boxed{h(t)} \longrightarrow \sum_{k=-\infty}^{\infty} A_k |H(\omega_k)| \cos(\omega_k t + \theta_k + \angle H(\omega_k))$$

- Def'n of FT came from passing sinewave thru LTI System ⇒ lead to Fourier Transform of Imp se Response

$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega)$: characterizes response of LTI system as a function of frequency

- THEN used same definition for signal, but interpret differently:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$x(t)$ — time-domain

$X(\omega) = \mathcal{F}\{x(t)\}$

$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$

$X(\omega)$ — frequency-domain

$$x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$= \mathcal{F}^{-1}\{X(\omega)\}$$

- $X(\omega)$, or more specifically $|X(\omega)|^2$, characterizes the energy distribution of $x(t)$ as a function of frequency

Recall, Parseval's Theorem:

$$\underset{\text{energy}}{E_x} = \int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

Note: Chap. 5 Discrete Time Fourier Transform (DTFT)

- $$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

- Similar to CT, the Definition of the DTFT comes from passing DT sinewave thru DT LTI System:

$$x[n] = e^{\omega_o n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

$\forall n$

$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega_o (n-k)}$$

$$= e^{j\omega_o n} \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_o k}$$

$$= H(\omega_o)\, e^{j\omega_o n}$$

- where:

$$h[n] \xleftrightarrow{DTFT} H(\omega) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

- characterizes how DT LTI system responds as a function of frequency BUT remember DT frequencies are only unique over a 2π interval (e.g., $-\pi < \omega < \pi$)
- For DT signals: $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$