Convolution of Two Gaussian Pulses

$$\frac{1}{\sigma_1\sqrt{2\pi}}\, e^{-\frac{t^2}{2\sigma_1^2}} * \frac{1}{\sigma_2\sqrt{2\pi}}\, e^{-\frac{t^2}{2\sigma_2^2}}$$

$$= \frac{1}{\sigma_3\sqrt{2\pi}}\, e^{-\frac{t^2}{2\sigma_3^2}} \qquad \sigma_3^2 = \sigma_1^2 + \sigma_2^2$$

And, again, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{t^2}{2\sigma^2}}\, dt = 1$

Derivation: convolving two Gaussians

$$\int_{-\infty}^{\infty} e^{-\frac{\tau^2}{2\sigma_1^2}} e^{-\frac{(t-\tau)^2}{2\sigma_2^2}} d\tau$$

$$= e^{-\frac{t^2}{2\sigma_2^2}} \int_{-\infty}^{\infty} e^{\left(-\frac{\tau^2}{2\sigma_1^2} + \frac{t\tau}{\sigma_2^2} - \frac{\tau^2}{2\sigma_2^2}\right)} d\tau$$

Focus on exponent inside sum: $\left\{\begin{array}{l}\text{put over common} \\ \text{denominator } \sigma_1^2 \sigma_2^2\end{array}\right.$

$$-\frac{1}{2}\frac{(\sigma_1^2+\sigma_2^2)}{\sigma_1^2\sigma_2^2}\left\{\tau^2 - \left(\frac{2\sigma_1^2 t}{\sigma_1^2+\sigma_2^2}\right)\tau + \underbrace{\frac{\sigma_1^4 t^2}{(\sigma_1^2+\sigma_2^2)^2}}_{\text{complete square}}\right\}$$

$$= -\frac{1}{2}\frac{(\sigma_1^2+\sigma_2^2)}{\sigma_1^2\sigma_2^2}\left\{\left(\tau - \frac{\sigma_1^2 t}{\sigma_1^2+\sigma_2^2}\right)^2\right\}$$

. have to multiply outside of integral by: $e^{\frac{1}{2}\frac{(\sigma_1^2+\sigma_2^2)}{\sigma_1^2\sigma_2^2}\frac{\sigma_1^4 t^2}{(\sigma_1^2+\sigma_2)^2}}$

At this point, we have:

$$e^{-\frac{t^2}{2\sigma_2^2}} \, e^{\frac{1}{2}\frac{\sigma_1^2 t^2}{\sigma_2^2(\sigma_1^2+\sigma_2^2)}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(\tau - \frac{\sigma_1^2 t}{\sigma_1^2+\sigma_2^2}\right)^2} d\tau}_{\substack{\text{area} = \sqrt{2\pi}\,\sigma \\ \text{Gaussian distribution}}}$$

where:

$$\sigma^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

Thus:

$$e^{-\frac{t^2}{2}\left\{\frac{\sigma_1^2+\sigma_2^2-\sigma_1^2}{\sigma_2^2(\sigma_1^2+\sigma_2^2)}\right\}} = e^{-\frac{t^2}{2\sigma_3^2}}$$

where:

$$\sigma_3^2 = \sigma_1^2+\sigma_2^2$$

Throwing in original multiplicative scalars:

$$\frac{1}{\sqrt{2\pi}\,\sigma_1} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_2} \cdot \sqrt{2\pi}\,\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}} = \frac{1}{\sqrt{2\pi}\,\sigma_3}$$

This completes proof.

- We know from Time-Invariance

$$\frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(t-t_1)^2}{2\sigma_1^2}} * \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(t-t_2)^2}{2\sigma_2^2}}$$

$$= \frac{1}{\sigma_3\sqrt{2\pi}} e^{-\frac{(t-(t_1+t_2))^2}{2\sigma_3^2}}$$

where: $\sigma_3^2 = \sigma_1^2 + \sigma_2^2$ Variances add

two :)