Symmetry Properties of Fourier Transform

FT: $$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- Consider $x(t)$ real-valued s.t. $x^*(t) = x(t)$ and take conjugate of both sides of FT eqn.

$$X^*(\omega) = \int_{-\infty}^{\infty} x^*(t) e^{j\omega t} dt$$

- Now, replace ω by $-\omega$ (and invoke $x^*(t) = x(t)$)

$$X^*(-\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X^*(-\omega) = X(\omega)$$

or $$X(-\omega) = X^*(\omega)$$

- Expressing in polar form : $X(\omega)=|X(\omega)|e^{j\angle X(\omega)}$

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$$X^*(-\omega)=|X(-\omega)|\,e^{-j\angle X(-\omega)}$$

we find that for $x(t)$ real-valued:

$|X(-\omega)|=|X(\omega)| \Rightarrow$ magnitude is even-symmetric

$\angle X(-\omega)=-\angle X(\omega) \Rightarrow$ phase is odd-symmetric

See Example 4.1 on pp. 290-291 of text

Fig. 4.5 (a) $|X(\omega)|$ Fig. 4.5 (b) $\angle X(\omega)$

for: $x(t)=e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)=\dfrac{1}{a+j\omega}$

- **Recall:** $x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

- Holds for $a=-1$: $x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-\omega)$

- Now, consider $x(t)$ real & even-symmetric

$$x(-t) = x(t)$$

- Take FT of both sides:

$$X(-\omega) = X(\omega)$$

- Since $x(t)$ real-valued:

$$X^*(\omega) = X(\omega) \quad \text{for all } \omega$$

- can only be true if $X(\omega)$ is real-valued

$x(t)$ real-valued, even-symmetric $\overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ real-valued, even-symmetric

- Consider $x(t)$ real-valued & odd-symmetric

$$x(-t) = -x(t)$$

- Take FT of both sides:

$$X(-\omega) = -X(\omega)$$

- Since $x(t)$ real-valued:

$$X^*(\omega) = -X(\omega)$$

- can only be true if $X(\omega)$ is purely imaginary
  $\Rightarrow$ no real-part

$x(t)$ real-valued, odd-symmetric $\overset{\mathcal{F}}{\longleftrightarrow}$ $X(\omega)$ purely imaginary, odd-symmetric

- Note: consider $x(t)$ real & even such that $X(\omega)$ real & even

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} X(\omega)\, e^{-j\omega t_0}$$

If you shift in time so that $x(t-t_0)$ is not even-symmetric (about $t=0$), the FT of $x(t-t_0)$ becomes complex-valued