

Fourier Transform

①

- Consider sinewave input to LTI System

$$x(t) = e^{j\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = ? = H(\omega_0) e^{j\omega_0 t}$$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) e^{j\omega_0(t-\tau)} d\tau \\ &= \left\{ \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0\tau} d\tau \right\} e^{j\omega_0 t} = H(\omega_0) e^{j\omega_0 t} \end{aligned}$$

where $H(\omega)$ is the Fourier Transform of the impulse response defined as

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

(τ was just a dummy variable of integration)

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The FT of a signal $x(t)$ is defined the same way, just interpreted differently:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

• $X(\omega)$ reflects how the energy of a signal is distributed as a function of frequency

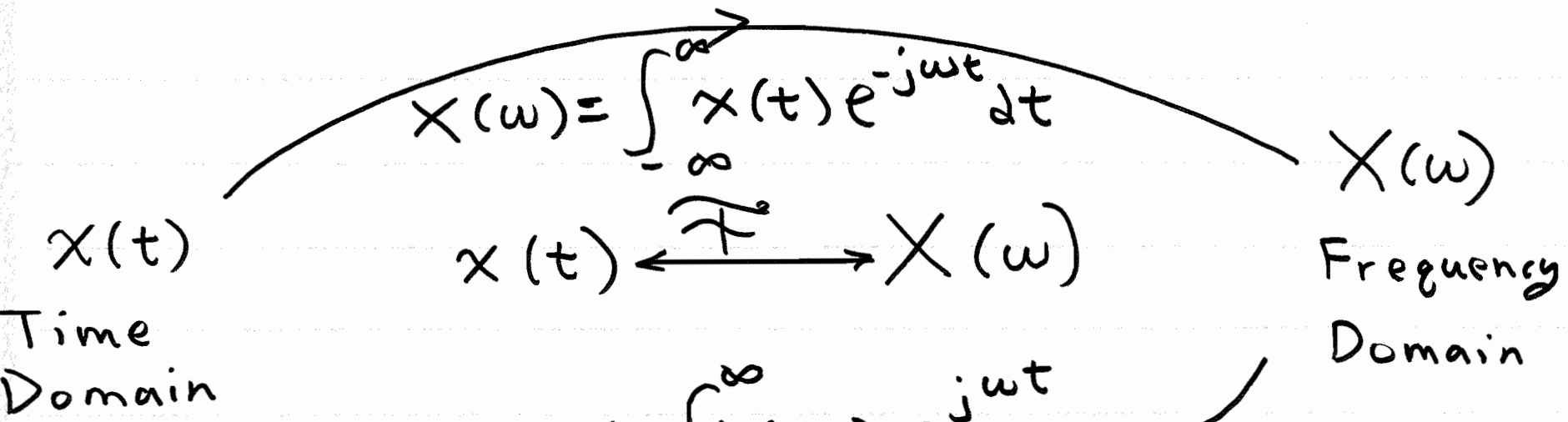
• we will prove Parseval's Theorem later

$$\text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

• other useful info. about the signal is gleaned from $X(\omega)$ which is generally complex-valued for every frequency ω

• Notation and Inverse Fourier Transform

$$X(\omega) = \mathcal{F}\{x(t)\}$$



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$

• VIP: this mapping is one-to-one \Rightarrow uniqueness

Notation Issue:

④

Recall Laplace Transform: $X(s) = \int_0^{\infty} x(t) e^{-st} dt$

- One-Sided Laplace Transform useful for dealing with initial conditions
- Two-Sided Laplace Transform defined as

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

- Thus, many textbooks make connection between Laplace and Fourier Transforms as

$$X(j\omega) = X(s) \Big|_{s=j\omega} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- That is, the Fourier Transform is the Laplace Transform evaluated along the imaginary axis in the s-plane $\Rightarrow s = \sigma + j\omega$

• We will not use this notation

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• it's awkward, confusing, and cumbersome

• Consider basic Fourier Transform in Example 4.4 on pg. 293 for rectangular pulse

$$x(t) = \text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{T} X(\omega) = \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$
$$= u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)$$

• The book writes: $X(j\omega) = \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$

• does that make sense?

$\frac{\omega}{2}$
no j here

• Also, we will define the Fourier Transform of signals, like sine waves turned-on forever, for which the Laplace Transform is not even defined

$$\bullet X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = |X(\omega)| e^{j\angle X(\omega)} \quad (6)$$

is generally complex-valued at every frequency ω

• The phase of $X(\omega)$, $\angle X(\omega)$, allows us to distinguish between the FT of $x(t)$ and the FT of $x(t-t_0)$ } energy distribution as a function of ω is same for both

• The phase of $X(\omega)$ also allows us to distinguish between the FT of $\cos(\omega_0 t)$ and the FT of $\sin(\omega_0 t)$ } both have all their energy concentrated at ω_0 and $-\omega_0$

• Remember the uniqueness of the Fourier Transform 1-to-1 mapping between time domain and frequency domain

Example 4.1 on pg. 290 of text

⑦

$$x(t) = e^{-at} u(t)$$

$$X(\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$= \left. \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \right|_0^{\infty} = \frac{-1}{a+j\omega} (0-1)$$

• have our first FT pair:

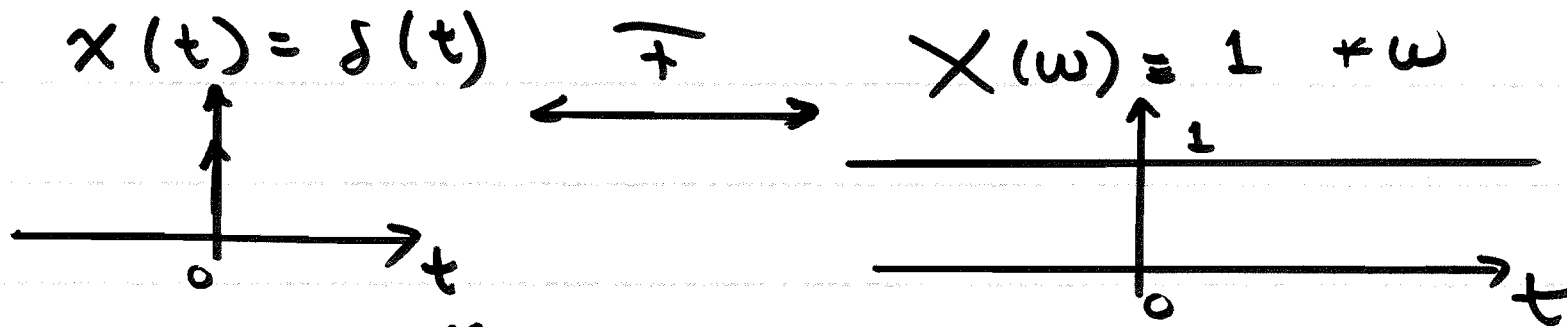
$$x(t) = e^{-at} u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega} = X(\omega)$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}} \quad ; \quad \angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

See Plots in Fig. 4.5 on pg. 291

• Example 4.3

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Proof:
$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^0 dt = 1$$

• An impulse (Dirac Delta) has equal energy at any and all frequencies

• Intuition: narrow in time-domain

⇒ wide in frequency domain
(take up a lot of bandwidth)

• this is the extreme case

- What about $x(t) = 1 \quad \forall t \Rightarrow$ DC $\omega=0$
- conjecture $X(\omega) \propto \delta(\omega)$ since all energy is concentrated at $\omega=0 \Rightarrow$ no other frequency content

• validate by using inverse Fourier Transform

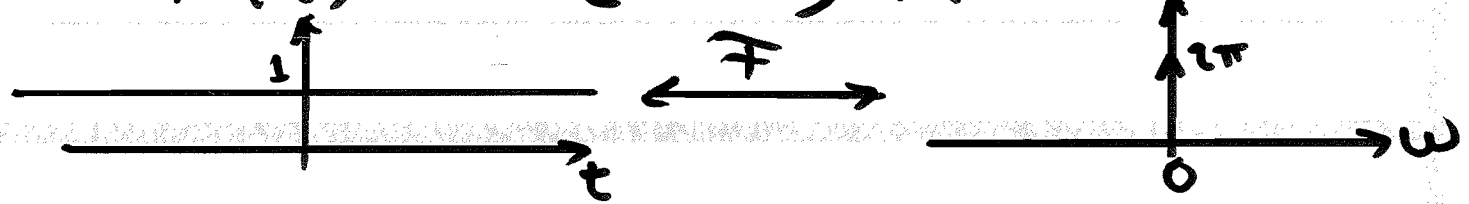
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

• Try $X(\omega) = \alpha \delta(\omega) : \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha \delta(\omega) e^{j\omega t} d\omega$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha \delta(\omega) e^{j0} d\omega = \frac{\alpha}{2\pi}$$

• (choosing $\alpha=2\pi$, we get

$$x(t) = 1 \xleftrightarrow{\mathcal{F}} X(\omega) = 2\pi \delta(\omega)$$



• Sine waves

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$$x(t) = e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} X(\omega) = ?$$

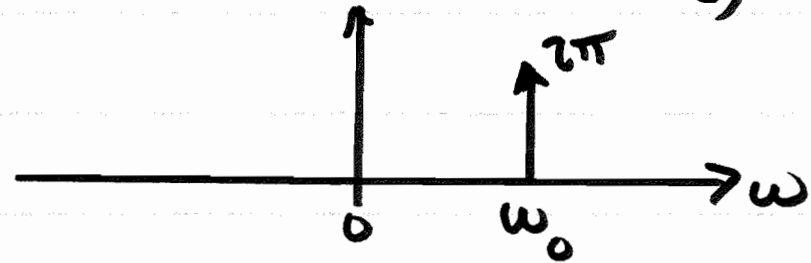
for all t

- only frequency present is $\omega_0 \Rightarrow$
all energy concentrated at $\omega_0 \Rightarrow$
conjecture $X(\omega) = 2\pi \delta(\omega - \omega_0)$

- check inverse Fourier Transform:

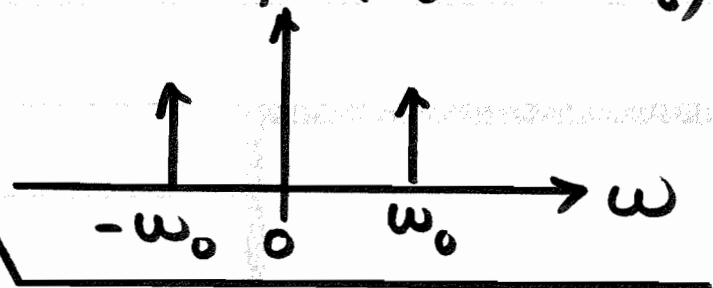
$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{\infty} (2\pi \delta(\omega - \omega_0)) e^{j\omega t} d\omega \\ &= \frac{2\pi}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega_0 t} d\omega \\ &= \left\{ \int_{-\infty}^{\infty} \delta(\omega - \omega_0) d\omega \right\} e^{j\omega_0 t} = e^{j\omega_0 t} \end{aligned}$$

$$x(t) = e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} X(\omega) = 2\pi \delta(\omega - \omega_0) \quad (11)$$



• Using Euler's formulas plus the fact that the Fourier Transform is a linear operator, we have

$$\cos(\omega_0 t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} \xleftrightarrow{\mathcal{F}} \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$$

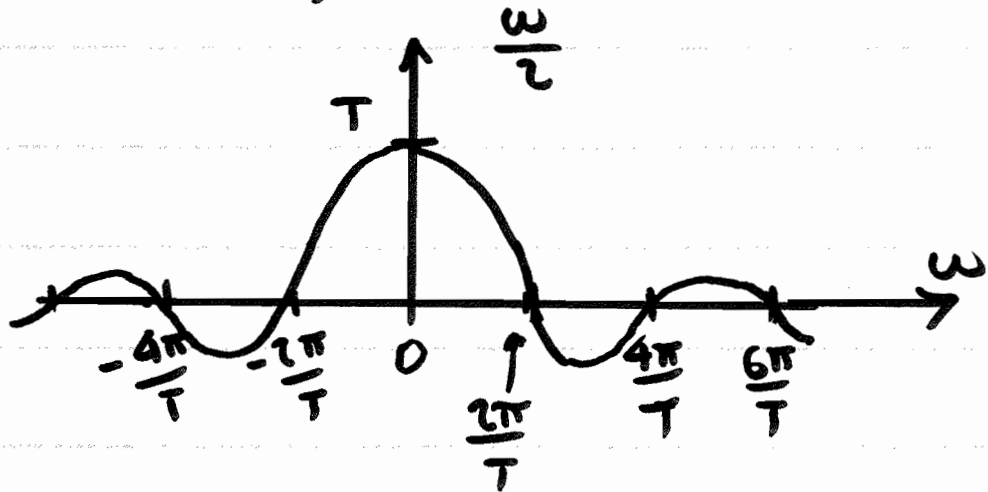
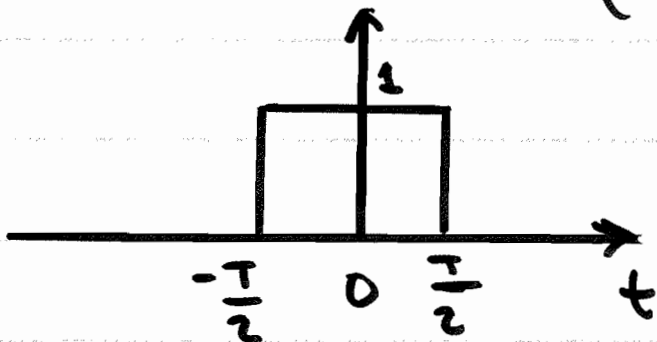


$$\sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} \xleftrightarrow{\mathcal{F}} \frac{\pi}{j} \delta(\omega - \omega_0) - \frac{\pi}{j} \delta(\omega + \omega_0)$$

Example 4.4

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$$x(t) = \text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{F} X(\omega) = \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$



Proof on next page:

- for value at $\omega=0$, use L'Hospital's Rule:

$$X(\omega)|_{\omega=0} = \frac{\frac{T}{2} \cos\left(T\frac{\omega}{2}\right)|_{\omega=0}}{\frac{1}{2}} = T$$

- zero crossings at: $T\frac{\omega}{2} = m\pi$, $\sin(m\pi) = 0$
for m integer

- $X(\omega) \rightarrow 0$ as $\omega \rightarrow \infty$
or $\omega \rightarrow -\infty$

• Proof: $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ (13)

$$= \int_{-T/2}^{T/2} e^{-j\omega t} dt = \left[\frac{-1}{j\omega} e^{-j\omega t} \right]_{-T/2}^{T/2}$$

$$= \frac{-1}{j\omega} \left\{ e^{-j\omega \frac{T}{2}} - e^{j\omega \frac{T}{2}} \right\} \frac{2}{2}$$

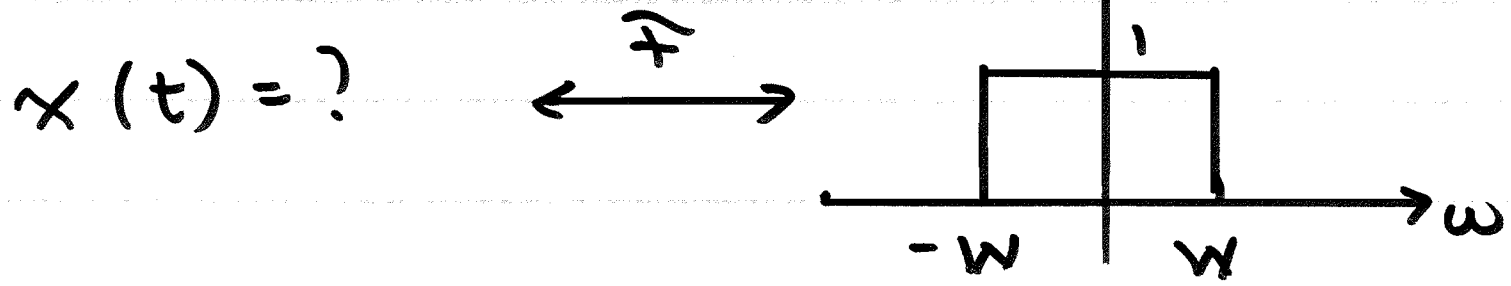
$$= \frac{2}{2j} \left\{ \frac{1}{2j} e^{j\omega \frac{T}{2}} - \frac{1}{2j} e^{-j\omega \frac{T}{2}} \right\}$$

$$= \frac{\sin\left(T \frac{\omega}{2}\right)}{\omega/2}$$

} purely
real-valued
because $x(t)$ is symmetric
 $x(-t) = x(t)$

Example 4.5 What if rectangle shape
is in frequency domain? Lowpass Filter

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Use inverse Fourier Transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-W}^W e^{j\omega t} d\omega = \frac{1}{2\pi j t} \left\{ e^{j\omega t} \right\}_{-W}^W$$

$$= \frac{1}{\pi t} \frac{1}{2j} \left\{ e^{j\omega t} - e^{-j\omega t} \right\} = \frac{\sin(Wt)}{\pi t}$$

