

• consider: $x(t) = x(t+T)$ for all t

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• Fourier series expansion for periodic signal with period T :

$$x(t) = \sum_{k=-\infty}^{\infty} a_k s_k(t)$$

$$s_k(t) = e^{j 2\pi \frac{k}{T} t}$$

"Fourier Series coefficients"

= complex amplitudes of the sinewaves

orthogonal sinewaves forming a complete basis for all periodic signals with period T

$\frac{1}{T}$ = fundamental frequency in Hz

$\frac{k}{T}$ = k -th harmonic frequency (k = integer)

• $s_k(t)$ is periodic with period $\frac{T}{k}$ and is thus periodic with period T (k = integer)

• $s_k(t)$ and $s_l(t)$ are orthogonal:

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$$\int_{-T/2}^{T/2} s_k(t) s_l^*(t) dt = 0 \quad \text{if } k \neq l$$

Proof:
$$\int_{-T/2}^{T/2} e^{j 2\pi \frac{k}{T} t} e^{-j 2\pi \frac{l}{T} t} dt = \int_{-T/2}^{T/2} e^{j 2\pi \frac{(k-l)}{T} t} dt$$

$$= \left[\frac{1}{j 2\pi \frac{(k-l)}{T}} e^{j 2\pi \frac{(k-l)}{T} t} \right]_{-T/2}^{T/2}$$

$$= \frac{T}{j 2\pi (k-l)} \left\{ e^{j 2\pi \frac{(k-l)}{T} \frac{T}{2}} - e^{-j 2\pi \frac{(k-l)}{T} \frac{T}{2}} \right\}$$

$$= \frac{T}{j 2\pi (k-l)} \left\{ (e^{j \pi})^{k-l} - (e^{-j \pi})^{k-l} \right\} = 0$$

since $e^{j \pi} = e^{-j \pi} = -1$

More generally:
$$\int_{t_0}^{t_0+T} s_k(t) s_l^*(t) dt = T \delta[k-l]$$

$$t_0 \text{ arbitrary}$$

e.g.
$$\int_0^T s_k(t) s_l^*(t) dt = \begin{cases} T, & \text{if } k=l \\ 0, & \text{if } k \neq l \end{cases} = T \delta[k-l]$$

- As a result, the FS coefficients (complex amplitudes) may be found as:

$$a_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) s_k^*(t) dt = \frac{1}{T} \int_{t_0}^{t_0+T} \left(\sum_{l=-\infty}^{\infty} a_l s_l(t) \right) s_k^*(t) dt$$

$$= \sum_{l=-\infty}^{\infty} a_l \int_{t_0}^{t_0+T} s_l(t) s_k^*(t) dt = \sum_{l=-\infty}^{\infty} a_l \frac{1}{T} \cdot T \delta[k-l] = a_k$$

- in particular:

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j 2\pi \frac{k}{T} t} dt$$

Sidenote:

Recall energy of a signal: $E_x = \int_{-\infty}^{\infty} x^2(t) dt$

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For more general case of a complex-valued signal:

NOTATION

$$x(t) = x_I(t) + j x_Q(t)$$

in-phase \nearrow \nwarrow quadrature-phase

OR: $x(t) = x_r(t) + j x_i(t)$

real part =

$$\operatorname{Re}\{x(t)\}$$

$$= \frac{1}{2} \{x(t) + x^*(t)\}$$

imaginary part

$$\operatorname{Im}\{x(t)\}$$

$$= \frac{1}{2j} \{x(t) - x^*(t)\}$$

$$\text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x_r^2(t) dt + \int_{-\infty}^{\infty} x_i^2(t) dt$$

= energy of real part + energy of imaginary part

• Energy for sinewave is typically computed as energy per period:

$$E_R = \frac{1}{T} \int_{-T/2}^{T/2} |s_R(t)|^2 dt$$

$$|s_R(t)|^2 = s_R(t) s_R^*(t)$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} e^{j 2\pi \frac{R}{T} t} e^{-j 2\pi \frac{R}{T} t} dt = \frac{1}{T} \int_{-T/2}^{T/2} e^0 dt = 1$$

• With a complex-amplitude:

energy of $a_R s_R(t)$ is: $|a_R|^2$ } $a_R e^{j 2\pi \frac{R}{T} t}$ = energy of

Critical Property \Rightarrow Parseval's Theorem

• Energy per period of periodic signal = sum of energy (per period) in each of the harmonic sinewaves in the FS expansion of $x(t)$

= energy in k -th harmonic term in FS expansion with frequency $\frac{k}{T}$ in Hz

• Mathematically:

$$\frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

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• Proof derives from orthogonality of the sinewaves:

$$\frac{1}{T} \int_{-T/2}^{T/2} x(t) x^*(t) dt = \frac{1}{T} \int_{-T/2}^{T/2} \sum_k a_k s_k(t) \sum_l a_l^* s_l^*(t) dt$$

$$= \sum_k \sum_l a_k a_l^* \underbrace{\frac{1}{T} \int_{-T/2}^{T/2} s_k(t) s_l^*(t) dt}_{\frac{1}{T} \cdot T \delta[k-l]} = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Q.E.D.

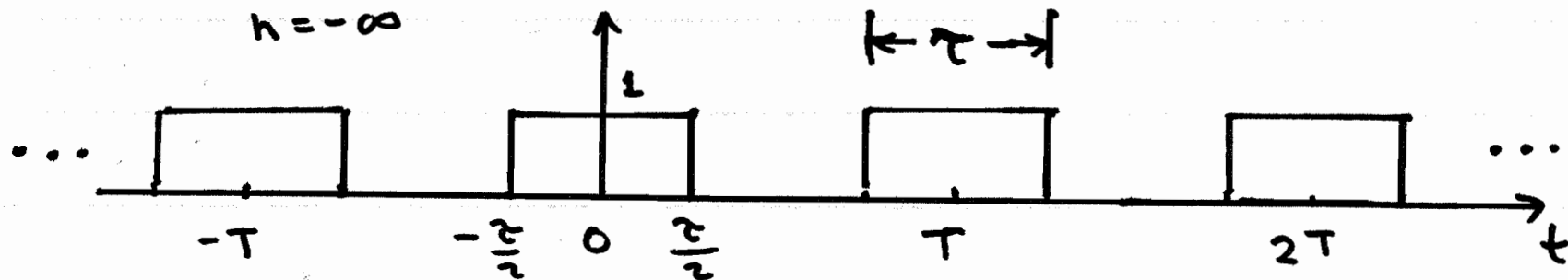
• This is our first (big) step towards examining how the energy of a signal is distributed as a function of frequency!

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• Some basic Fourier Series!

1. Periodic train of rectangular pulses:

$$x(t) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t-nT}{\tau}\right)$$



$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j2\pi \frac{k}{T} t} dt = \frac{1}{T} \int_{-\tau/2}^{\tau/2} e^{-j2\pi \frac{k}{T} t} dt$$

$$= \frac{1}{T} \cdot \frac{1}{-j2\pi \frac{k}{T}} \left[e^{-j2\pi \frac{k}{T} t} \right]_{-\tau/2}^{\tau/2} = \frac{1}{k\pi} \frac{-1}{2j} \left\{ e^{-j\frac{2\pi k \tau}{2T}} - e^{j\frac{2\pi k \tau}{2T}} \right\}$$

$$a_k = \frac{\sin\left(k\pi \frac{\tau}{T}\right)}{k\pi} \quad -\infty < k < \infty$$

at $k=0 \Rightarrow \frac{\tau}{T}$

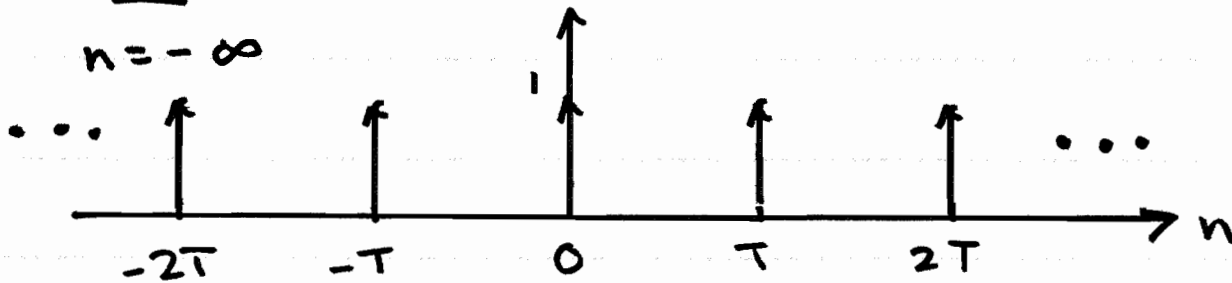
See Example 3.5 ch pg. 193-195

2. Periodic Train of Dirac Delta Functions

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⇒ very important for our theoretical analysis
of sampling theory later in Chap. 7

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j 2\pi \frac{k}{T} t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-j 2\pi \frac{k}{T} t} dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^0 dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) dt = \frac{1}{T} \quad \forall k$$

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{-j 2\pi \frac{k}{T} t}$$

Some Key Properties of Fourier Series

• Starting point: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{+j 2\pi \frac{k}{T} t}$

• Time-Shift Property:

$$y(t) = x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k e^{j 2\pi \frac{k}{T} (t-t_0)}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\left(a_k e^{j 2\pi \frac{k}{T} t_0} \right)}_{\text{new FS coeffs for } y(t) = x(t-t_0)} e^{j 2\pi \frac{k}{T} t}$$

new FS coeffs for $y(t) = x(t-t_0)$

• Time-Scaling:

$$y(t) = x(at) = \sum_{k=-\infty}^{\infty} a_k e^{+j 2\pi \frac{k}{T} at} = \sum_{k=-\infty}^{\infty} a_k e^{j 2\pi \frac{k}{T/a} t}$$

⇒ FS coefficients are the same

⇒ new period = T/a

• Differentiation

$$y(t) = \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} \underbrace{(a_k j 2\pi \frac{k}{T})}_{\text{new FS coeffs. for } y(t) = \frac{dx(t)}{dt}} e^{j 2\pi \frac{k}{T} t}$$

• For additional properties of FS, see Table 3.1 on pg. 206

• Basic Property: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j 2\pi \frac{k}{T} t}$ $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j 2\pi \frac{k}{T} t}$

$$z(t) = Ax(t) + By(t)$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{(Aa_k + Bb_k)}_{\text{new FS coeffs. for } z(t)} e^{j 2\pi \frac{k}{T} t}$$

same period