SOLUTION

Name:

Final Exam

ECE301 Signals and Systems Wednesday, May 6, 2015

Cover Sheet

Write your name on this page and every page to be safe.

Test Duration: 120 minutes.

Coverage: Comprehensive

Open Book but Closed Notes. Three two-sided handwritten sheets.

Calculators NOT allowed.

This test contains **four** problems, each with multiple parts.

You have to draw your own plots.

You must show all work for each problem to receive full credit.

Good luck! It was great having you in class this semester!

Have a great Summer!

Problem 1. Using the Z-Transform, determine the output of the system

$$y[n] = \alpha y[n-1] + x[n]$$

for the input $x[n] = \beta^n u[n]$. You must use the Z-Transform to solve this problem, i.e., use basic Z-Transform properties and a basic Z-Transform pair plus partial fraction expansion to ultimately determine the output $y[n]$. You are given that $|\alpha| < 1$ and $|\beta| < 1$.

ZT of Difference Eqn. $\Rightarrow$ both sides

$$Y(z) = \alpha z^{-1} Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$$

$$X(z) = \frac{z}{z-\beta} \Rightarrow Y(z) = H(z)X(z) = \frac{z^2}{(z-\alpha)(z-\beta)}$$

$$z^{-1}Y(z) = \frac{z}{(z-\alpha)(z-\beta)} = \frac{A_1}{z-\alpha} + \frac{A_2}{z-\beta} \qquad 1$$

$$A_1 = (z-\alpha)\frac{z}{(z-\alpha)(z-\beta)}\bigg|_{z=\alpha} = \frac{\alpha}{\alpha-\beta}$$

$$A_2 = (z-\beta)\frac{z}{(z-\alpha)(z-\beta)}\bigg|_{z=\beta} = \frac{\beta}{\beta-\alpha}$$

Now multiply by z on both sides of 1

$$Y(z) = \frac{\cancel{\alpha}}{\alpha-\beta}\,\frac{z}{z-\alpha} + \frac{\beta}{\beta-\alpha}\,\frac{z}{z-\beta}$$

$$y[n] = \frac{\alpha}{\alpha-\beta}\alpha^n u[n] + \frac{\beta}{\beta-\alpha}\beta^n u[n]$$

which agrees with what we derived in class back in Chap. 2 via DT convolution

Problem 2.
The rectangular pulse $x_{in}(t) = \{u(t) - u(t-1)\}$ of duration 1 sec is input to the following integrator

$$x_a(t) = 3\int_{t-2}^{t} x_{in}(\tau)d\tau - 3\int_{t-5}^{t-3} x_{in}(\tau)d\tau$$

The output $x_a(t)$ is sampled every $T_s = 1/3$ seconds to form $x[n] = x_a(nT_s)$. The sampling rate is $f_s = 3$ samples/sec. Show work. Clearly label and write your final answer in the space provided on the next few pages.

(a) Do a stem plot of the DT signal $x[n]$, or you can simply write the numbers that comprise $x[n]$ (indicate with an arrow where the $n = 0$ value is.)

(b) The Discrete-Time (DT) signal $x[n]$, created as described above, is input to the DT system described by the difference equation below:

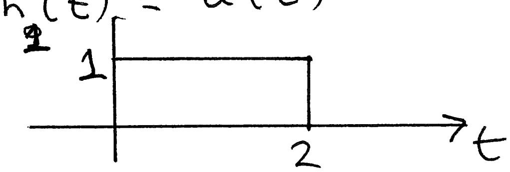
$$y[n] = x[n] - x[n-1]$$

(i) First, determine the impulse response $h[n]$ for this system.

(ii) Determine and plot the output $y[n]$ by convolving the input $x[n]$ defined above with the impulse response $h[n]$. Show all work in the space provided. ~~Do the stem plot for $y[n]$ on the graph provided on the page after next.~~ Sequence Form

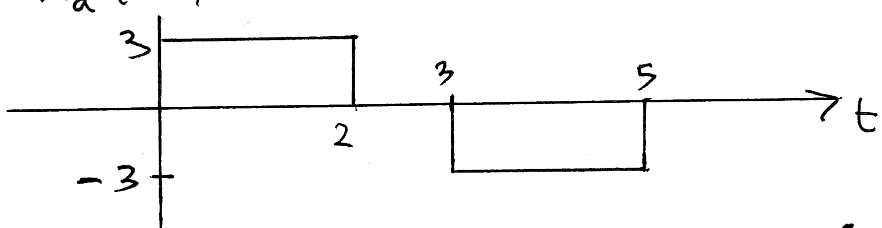
Impulse response of $y_1(t) = \int_{t-2}^{t} x_{in}(\tau)\,d\tau$

is $h_1(t) = u(t) - u(t-2)$

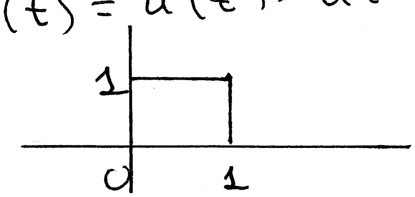


System 2 is simply
System 1 with t replaced by $t-3$
Thus: $h_2(t) = h_1(t-3)$ by Time Invariance

Thus: overall impulse response is:
$h_a(t) = 3h_1(t) - 3h_1(t-3)$

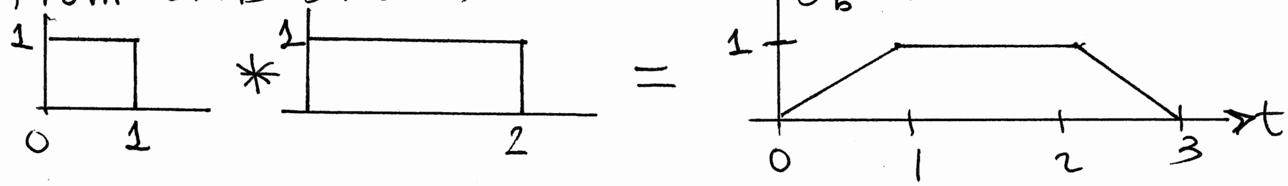


This is convolved with $x_{in}(t) = u(t) - u(t-1)$
to form
$x_a(t) = x_{in}(t) * h_a(t)$



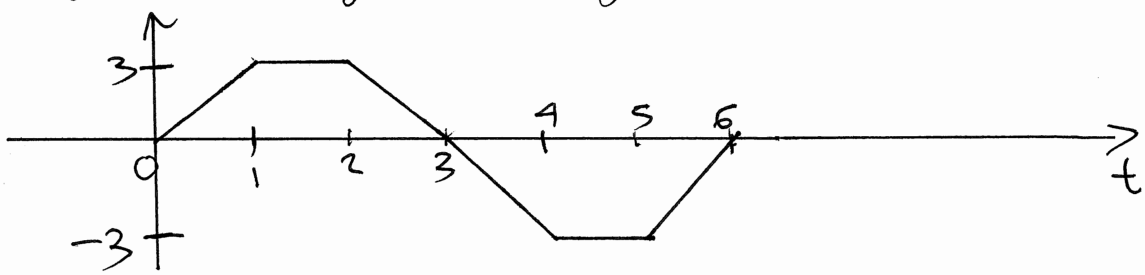
Show your work and plots for Problem 2 here.

From crib sheet, we know

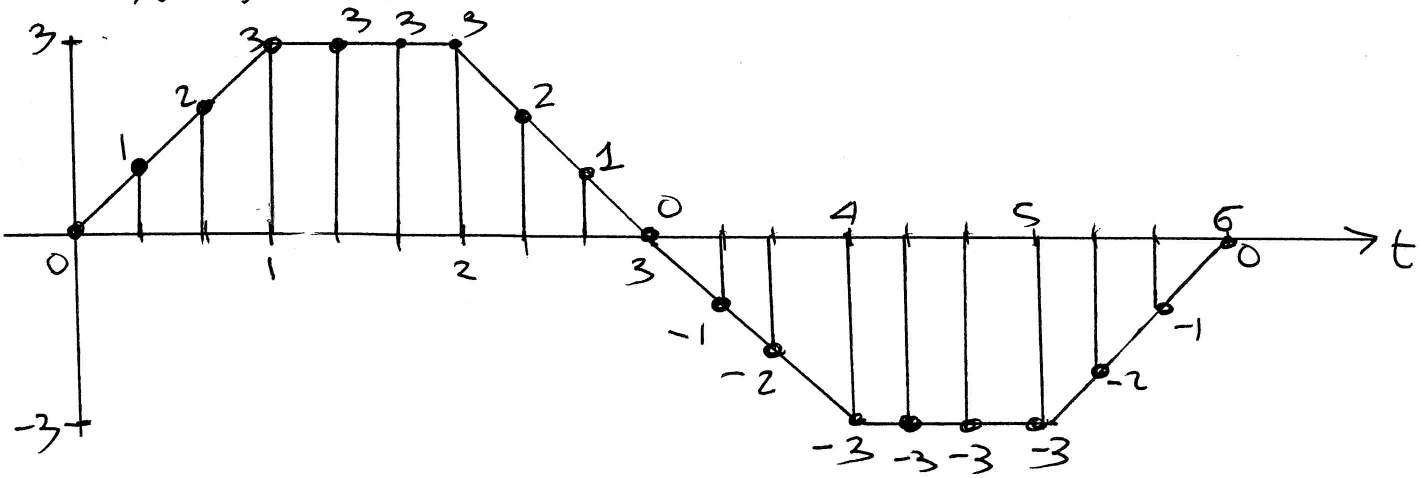


Thus, invoking Time-Invariance and Linearity:

$$x_a(t) = 3\,y_b(t) - 3\,y_b(t-3)$$



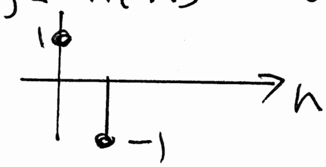
$x[n] = x_a(nT_s)$ $T_s = \frac{1}{3}$ $\Rightarrow$ so make plot larger



$$x[n] = \{\underset{\uparrow\, n=0}{0}, 1, 2, 3, 3, 3, 3, 2, 1, 0, -1, -2, -3, -3, -3, -3, -2, -1\}$$

(b) Impulse Response of $y[n] = x[n] - x[n-1]$

is $h[n] = \delta[n] - \delta[n-1]$



$y[n] = x[n] * h[n] = x[n] - x[n-1]$

$\{0, 1, 2, 3, 3, 3, 3, 2, 1, 0, -1, -2, -3, -3, -3, -3, -2, -1, 0\}$

$= \{0, 0, 1, 2, 3, 3, 3, 3, 2, 1, 0, -1, -2, -3, -3, -3, -3, -2, -1\}$

$= \{0, 1, 1, 1, 0, 0, 0, -1, -1, -1, -1, -1, -1, 0, 0, 0, 1, 1, 1\}$ ANS

Problem 3 (a). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. That is, the Fourier Transform of the analog signal $x_a(t)$ is exactly zero for $|\omega| > 40$ rads/sec. This signal is sampled at a rate $\omega_s = 100$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{100}$ sec. This yields the discrete-time sequence

$$x[n] = x_a(nT_s) = \left\{ \frac{\pi}{5} \frac{\sin\left(\frac{\pi}{10}n\right)}{\pi n T_s} \frac{\sin\left(\frac{3\pi}{10}n\right)}{\pi n T_s} \right\} 2j \sin\left(\frac{2\pi}{5}n\right) \quad \text{where:} \quad T_s = \frac{2\pi}{100}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \quad \text{where:} \quad T_s = \frac{2\pi}{100} \quad \text{and} \quad h(t) = T_s \frac{\pi}{10} \frac{\sin(10t)}{\pi t} \frac{\sin(50t)}{\pi t}$$

Problem 3 (b). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. This signal is sampled at the same rate $\omega_s = 100$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \quad \text{where:} \quad T_s = \frac{2\pi}{100} \quad \text{and} \quad h(t) = T_s \frac{1}{2} \left\{ \frac{\sin(40t)}{\pi t} + \frac{\sin(60t)}{\pi t} \right\}$$

Problem 3 (c). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. This signal is sampled at the same rate $\omega_s = 100$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \quad \text{where:} \quad T_s = \frac{2\pi}{100} \quad \text{and} \quad h(t) = T_s \frac{1}{2} \left\{ \frac{\sin(45t)}{\pi t} + \frac{\sin(55t)}{\pi t} \right\}$$

Show all your work for Prob. 3, parts (a)-(b)-(c) on next page.

Show your work for Prob. 3, parts (a)-(b)-(c) below.

Three criteria to check:

1. Is $\omega_s > 2\omega_M$?

2. Is $h(t)$ such that $H(\omega)=0$ for $|\omega| > \omega_s - \omega_M$?
This is necessary to filter out all the replicas

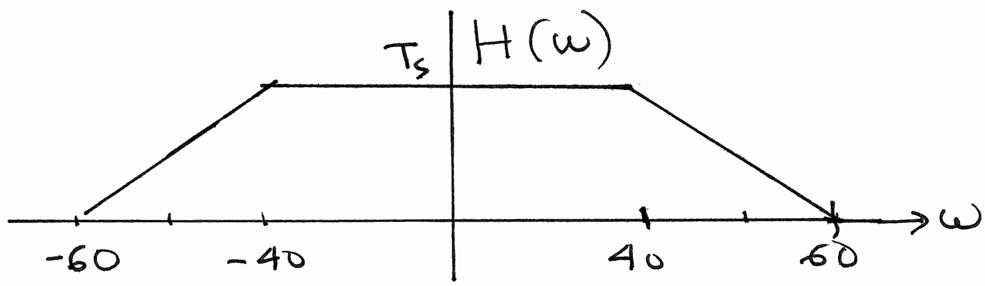
3. Is $h(t)$ such that $H(\omega) = T_s$ for $|\omega| < \omega_M$?
This insures that original Fourier Transform centered at $\omega=0$ is pass unaltered.

If these 3 criteria are met, then

$$x_r(t) = x_a(t) = x[n]\Big|_{n=t/T_s}$$

since $x[n] = x_a(t)\Big|_{t=nT_s}$

3(a) $\omega_s = 100$ $\quad \omega_M = 40 \Rightarrow 100 > 2(40) \Rightarrow$ no aliasing



Criteria 2+3 are met as well. THUS:

$$x_r(t) = x_a(t) = x[n]\Big|_{n=t/T_s} = \frac{100t}{2\pi} = \frac{50t}{\pi}$$

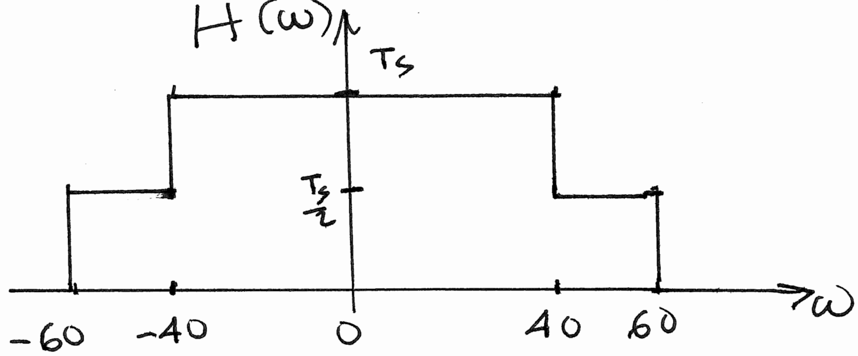
$$x_a(t) = \frac{\pi}{5}\frac{\sin\left(\frac{\pi}{10}\frac{50}{\pi}t\right)}{\pi t} \cdot \frac{\sin\left(\frac{3\pi}{10}\frac{50}{\pi}t\right)}{\pi t} \cdot 2j \sin\left(\frac{2\pi}{5}\cdot\frac{50}{\pi}t\right)$$

$$= \frac{\pi}{5}\frac{\sin(5t)}{\pi t} \cdot \frac{\sin(15t)}{\pi t} \cdot 2j \cdot \sin(20t)$$

3(b)

Show your work and plots for Problem ~~2~~ here.

3(b)



$H(\omega) = T_s$ for $|\omega| < 40$

$H(\omega) = 0$ for $|\omega| \geq \omega_s - \omega_M = 100 - 40 = 60$

$\omega_s = 100 > 2(40) \quad \Rightarrow x_r(t) = x_a(t)$

3(c) Same plot as above, except $40 \to 45$ and $60 \to 55$

Thus: $H(\omega) = T_s$ for $|\omega| < 40$

$= 0$ for $|\omega| > 60$

$\Rightarrow x_r(t) = x_a(t)$

Problem 3 (d). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. That is, the Fourier Transform of the analog signal $x_a(t)$ is exactly zero for $|\omega| > 40$ rads/sec. This signal is sampled at a rate $\omega_s = 80$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{80}$ sec. This yields the discrete-time sequence

$$x[n] = x_a(nT_s) = \left\{ \frac{\pi}{5} \frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n T_s} \frac{\sin\left(\frac{3\pi}{8}n\right)}{\pi n T_s} \right\} 2j \sin\left(\frac{\pi}{2}n\right) \qquad \text{where:} \quad T_s = \frac{2\pi}{80}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \qquad \text{where:} \quad T_s = \frac{2\pi}{80} \quad \text{and} \quad h(t) = T_s \frac{\sin(40t)}{\pi t}$$

Problem 3 (e). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. This signal is sampled at the same rate $\omega_s = 80$ rads/sec., but reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \qquad \text{where:} \quad T_s = \frac{2\pi}{80} \quad \text{and} \quad h(t) = T_s \left\{ \frac{\pi}{5} \frac{\sin(5t)}{\pi t} \frac{\sin(40t)}{\pi t} \right\}$$

Problem 3 (f). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. This signal is sampled at the same rate $\omega_s = 80$ rads/sec., but reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \qquad \text{where:} \quad T_s = \frac{2\pi}{80} \quad \text{and} \quad h(t) = T_s \frac{1}{2} \left\{ \frac{\sin(40t)}{\pi t} + \frac{\sin(10t)}{\pi t} \right\}$$

Show all your work for Prob. 3, parts (d)-(e)-(f) on next page.

Show your work for Prob. 3, parts (d)-(e)-(f) below.

3(d) $\omega_M = 40$ $\quad \omega_s = 80$ $\Rightarrow$ right at Nyquist Rate

$H(\omega) = T_s$ for $|\omega| < 40$ ✓

$H(\omega) = 0$ for $\underbrace{|\omega| > 40}$ ✓ replicas filtered out

$x_r(t) = x_a(t)$ $\quad \underset{\omega_s \nearrow \quad \nwarrow \omega_M}{80 - 40}$ $\quad \sin\left(\frac{\pi}{2}\,\frac{40t}{\pi}\right)$

$$= x[n]\Big|_{n = t/T_s} = \frac{\pi}{5}\,\frac{\sin\left(\frac{\pi}{8}\cdot\frac{40t}{\pi}\right)}{\pi t}\,\frac{\sin\left(\frac{3\pi}{8}\cdot\frac{40t}{\pi}\right)}{\pi t}\,2j\Big)$$

$$\underbrace{\frac{80t}{2\pi}} = \frac{40t}{\pi}\Bigg| = \frac{\pi}{5}\,\frac{\sin(5t)}{\pi t}\,\frac{\sin(15t)}{\pi t}\,2j \quad \sin(20t)$$

3(e) $\omega_s = 80$ $\quad \omega_M = 40$ $\Rightarrow$ ~~so~~ replicas are just touching each other

$\Rightarrow$ no room for roll-off of filter

$H(\omega) = T_s$ $\quad |\omega| < 35$ $\Big\}$ since $\omega_M = 40 \Rightarrow$ signal at will be altered

$H(\omega) = 0$ $\quad |\omega| > 45$ $\Big\}$ replicas start at $\omega_s - \omega_M = 80 - 40 = 40$

THUS, $x_r(t) \neq x_a(t)$

3(f) again, right at Nyquist Rate

$H(\omega) = T_s$ for $|\omega| < 10$ BUT $H(\omega) = \frac{T_s}{2}$ $\quad 10 \le |\omega| < 40$ ✗

$H(\omega) = 0$ for $|\omega| > 40$ ✓ replicas filtered out

$$x_r(t) \neq x_a(t)$$

Problem 3 (g). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. This signal is sampled at a rate $\omega_s = 50$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{50}$ sec, yielding the following discrete-time sequence:

$$x[n] = x_a(nT_s) = \left\{ \frac{\pi}{5} \frac{\sin\left(\frac{\pi}{5}n\right)}{\pi n T_s} \frac{\sin\left(\frac{3\pi}{5}n\right)}{\pi n T_s} \right\} 2j \sin\left(\frac{4\pi}{5}n\right) \qquad \text{where: } T_s = \frac{2\pi}{50}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a closed-form expression for the reconstructed signal $x_r(t)$. Show all work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \qquad \text{where: } T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s \frac{\sin(25t)}{\pi t}$$

Problem 3 (h). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. This signal is sampled at the same rate $\omega_s = 50$ rads/sec., where $\omega_s = 2\pi/T_s$ and the time between samples is $T_s = \frac{2\pi}{50}$ sec, but at a different starting point. This yields the Discrete-Time $x[n]$ signal below:

$$x_\epsilon[n] = x_a(nT_s + 0.5T_s) = \left\{ \frac{\pi}{5} \right\} \left\{ \frac{\sin(\frac{\pi}{5}(n+0.5))}{\pi(n+0.5)T_s} \frac{\sin(\frac{3\pi}{5}(n+0.5))}{\pi(n+0.5)T_s} \right\} 2j \sin\left(\frac{4\pi}{5}(n+0.5)\right)$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. *Hint:* before you do a lot of work, look at the interpolating lowpass filter being used below.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x_\epsilon[n] h(t - (n+0.5)T_s) \qquad \text{where: } T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s \frac{\sin(10t)}{\pi t}$$

Show your work for Prob. 3, parts (g)-(h) on next page.

Show all your work for Prob. 3, parts (g)-(h) on this page.

3 (g)

$\omega_M = 40 \qquad \omega_s = 50 < 2(40) \Rightarrow$ aliasing

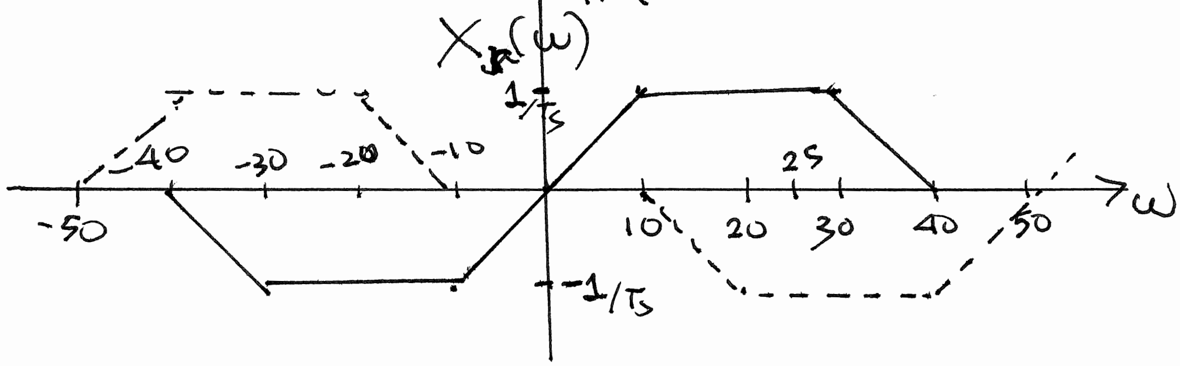
$H(\omega) = T_s$ for $|\omega| < 25$

$H(\omega) = 0$ for $|\omega| > 25$

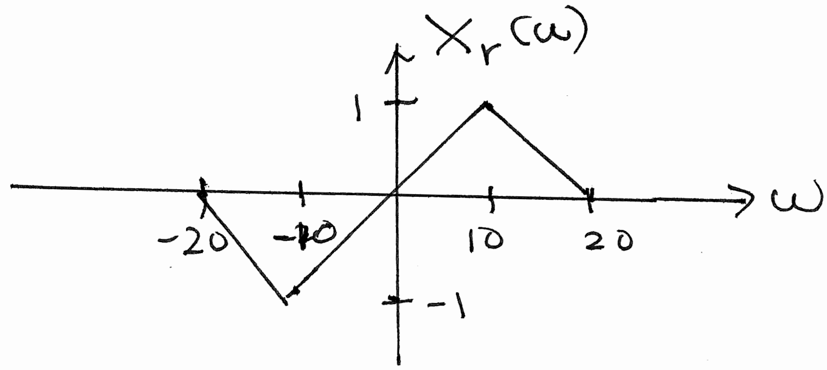
$$x_a(t) = x[n]\Big|_{n = t/T_s = \frac{50}{2\pi}t = \frac{25t}{\pi}}$$

$$= \frac{\pi}{5}\,\frac{\sin\left(\frac{\pi}{5}\frac{25}{\pi}t\right)}{\pi t}\,\frac{\sin\left(\frac{3\pi}{5}\frac{25}{\pi}t\right)}{\pi t}\,2j\sin\left(\frac{4\pi}{5}\frac{25}{\pi}t\right)$$

$$= \frac{\pi}{5}\,\frac{\sin(5t)}{\pi t}\,\frac{\sin(15t)}{\pi t}\,2j\sin(20t) = x_a(t)$$



Since $H(\omega) = 0$ for $|\omega| > 25$



$$x_r(t) = \frac{\pi}{5}\left\{\frac{\sin(5t)}{\pi t}\right\}^2 2j\sin(10t)$$

ANS:

THIS MEANS: $\quad x_r(t)\Big|_{t = n\frac{2\pi}{50}} = x_a(t)\Big|_{t = n\frac{2\pi}{50}}$

Show all your work for Prob. 3, parts (g)-(h) on this page.

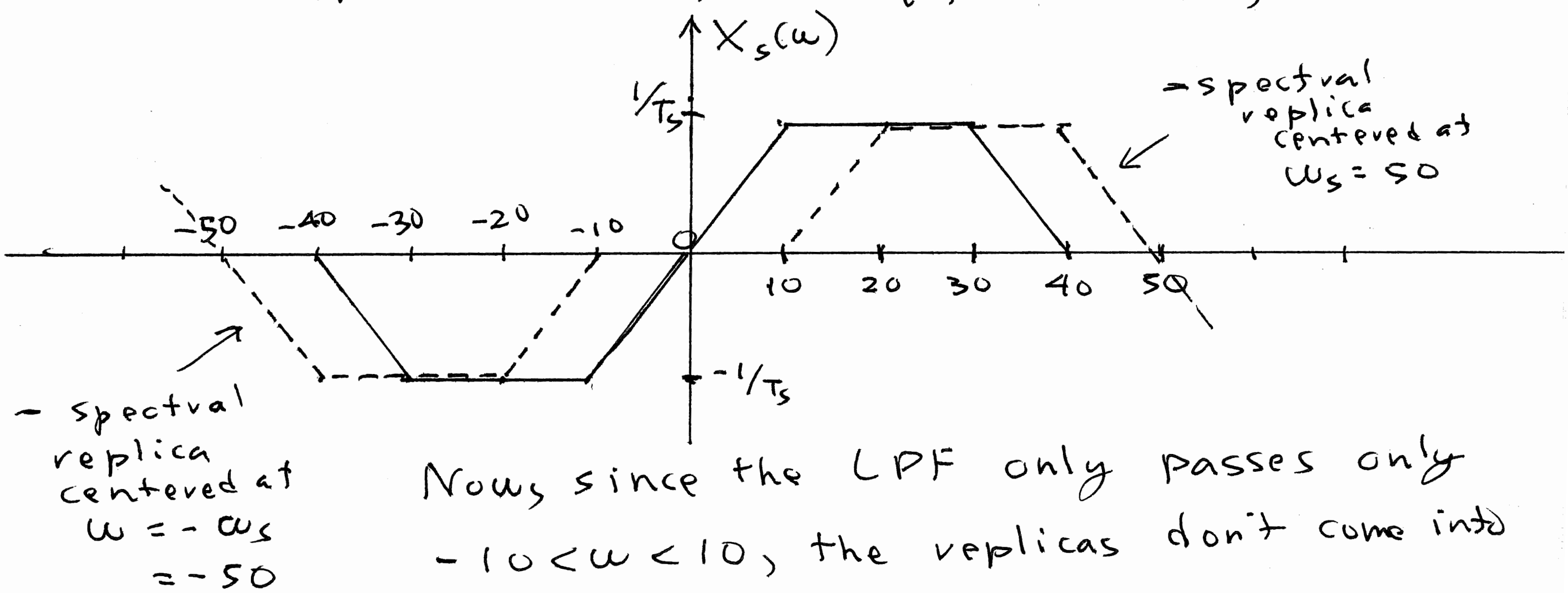
See handout called "Sample Time Invariance"

$$x_a(t)\,p(t-\tau) \xleftrightarrow{\mathcal{F}} \frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{-j2\pi k\frac{\tau}{T_s}}\, X_a\Big(\omega - k\underbrace{\tfrac{2\pi}{T_s}}_{\omega_s}\Big)$$

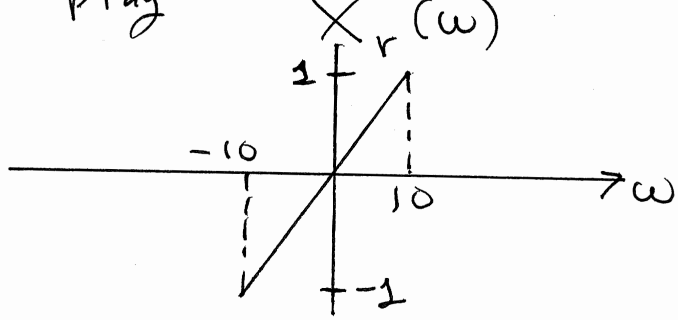
$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$

In this case $\tau = .5T_s$ $\Big\}$ $\displaystyle \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \underbrace{e^{-j2\pi k\frac{T_s}{2}\frac{1}{T_s}}}_{(-1)^k} X_a(\omega - k\omega_s)$

Thus, as opposed to the graph in 3(g), for 3(h), we have:



Now, since the LPF only passes only $-10<\omega<10$, the replicas don't come into play



$$x_r(t) = \frac{-j}{10}\frac{d}{dt}\left\{\frac{\sin(10t)}{\pi t}\right\}$$

Problem 4. For each part: show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer. Use the result below:

$$\int_{-\infty}^{\infty} e^{\frac{-(x-m)^2}{2\sigma^2}}\,dx = \sqrt{2\pi}\,\sigma$$

Problem 4 (a). The Gaussian pulse $x(t) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 3^2$, is multiplied by a complex-valued sinewave to form $y(t) = e^{j3t}x(t)$. Find the numerical value of $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega.$ $= \int_{-\infty}^{\infty} |y(t)|^2\,dt$ Parseval's Theorem

$|y(t)| = |e^{j3t}|\,|x(t)| = |x(t)| = x(t)$ since $x(t) > 0$ for all t

Thus: $\int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} x^2(t)\,dt$

$$= \frac{1}{\sigma_1^2\, 2\pi}\int_{-\infty}^{\infty} e^{-\frac{t^2 2}{2\sigma_1^2}}\,dt = \frac{1}{\sigma_1^2\, 2\pi}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2(\sigma_1/\sqrt{2})^2}}\,dt$$

$$= \frac{1}{\sigma_1^2\, 2\pi}\cdot\sqrt{2\pi}\,\frac{\sigma_1}{\sqrt{2}} = \frac{\sigma_1}{2\,\sigma_1^2\sqrt{\pi}} = \frac{1}{2\sqrt{\pi}\,\sigma_1} = \frac{1}{6\sqrt{\pi}}$$

Problem 4 (b). The Gaussian pulse $x(t) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 3^2$, is time-shifted to form $y(t) = x(t-3)$. Find the numerical value of $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega.$

$y(t) = x(t-3) \Rightarrow Y(\omega) = e^{-j3\omega}X(\omega)$

$|Y(\omega)| = |e^{-j3\omega}|\,|X(\omega)|$

$|Y(\omega)|^2 = |X(\omega)|^2$ + Parseval's Theorem

$E = \int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{6\sqrt{\pi}}$ from 4(a)

Problem 4 (c). The Gaussian pulse $x(t) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 3^2$, is differentiated to form $y(t) = \frac{d}{dt}x(t)$. Find the numerical value of $A = \int_{-\infty}^{\infty} y(t)dt$.

$$y(t) = \frac{d}{dt}x(t) \Rightarrow Y(\omega) = j\omega X(\omega)$$

$$A = \int_{-\infty}^{\infty} y(t)dt = Y(0) = j(0)X(0) = 0 \quad \text{ANS}$$

Also, $y(t) = \frac{d}{dt}x(t) = -\frac{2t}{2\sigma_1^2} e^{-\frac{t^2}{2\sigma_1^2}}$

$= \text{odd fn.} \times \text{even fn.} = \text{odd fn.}$

Area under odd function $= 0$

Problem 4 (d). The Gaussian pulse $x(t) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 3^2$, is multiplied by t to form $y(t) = tx(t)$. Find the numerical value of $A = \int_{-\infty}^{\infty} y(t)dt$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

$$y(t) = \underbrace{t}_{\text{odd fn.}}\ \underbrace{x(t)}_{\text{even fn.}} = \text{odd function}$$

Area under odd function $\int_{-\infty}^{\infty} = 0$ ANS

Problem 4 (e). The Gaussian pulse $x(t) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 3^2$, is compressed in time by a factor of 2 to form $y(t) = x(2t)$. Find the numerical value of the energy $E = \int_{-\infty}^{\infty} y^2(t)dt$.

$$y^2(t) = x^2(2t) = \left(\frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{2^2 t^2}{2\sigma_1^2}}\right)^2 = \frac{1}{\sigma_1^2(\sqrt{2\pi})^2} e^{-\frac{2^3 t^2}{2\sigma_1^2}}$$

$\sigma = \sigma_1$

$$= \frac{1}{\sigma_1^2 2\pi} e^{-\frac{t^2}{2\left(\frac{\sigma_1}{\sqrt{8}}\right)^2}}$$

$$E = \frac{1}{\sigma_1^2 2\pi} \sqrt{2\pi} \frac{\sigma_1}{\sqrt{8}} = \frac{1}{\sigma_1} \frac{1}{\sqrt{\pi}} \frac{1}{4} = \frac{1}{12\sqrt{\pi}}$$

Problem 4 (f). The Gaussian pulse $x(t) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 3^2$, is input to an LTI system with a Gaussian shaped impulse response, $h(t) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_2^2}}$, with $\sigma_2^2 = 4^2$. Determine a simple, closed-form expression for the output $y(t) = x(t) * h(t)$.

$$y(t) = x(t) * h(t) = \frac{1}{\sigma_3 \cdot \sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_3^2}}$$

where $\sigma_3^2 = 3^2 + 4^2 = 5^2$

$\sigma_3 = 5$

Problem 4 (g). The Gaussian pulse $x(t) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 3^2$, is input to an LTI system with a Gaussian shaped impulse response, $h(t) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_2^2}}$, with $\sigma_2^2 = 4^2$. Determine the numerical value of the area, $A = \int_{-\infty}^{\infty} y(t)dt$, under the output $y(t) = x(t) * h(t)$.

$$y(t) = x(t) * h(t) = \frac{1}{\sigma_3\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_3^2}} \qquad \sigma_3 = 5$$

$$\text{Area} = A = \int_{-\infty}^{\infty} y(t)\,dt = 1$$

Problem 4 (h). The Gaussian pulse $x(t) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 3^2$, is input to an LTI system with a Gaussian shaped impulse response, $h(t) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_2^2}}$, with $\sigma_2^2 = 4^2$. Determine the numerical value of the energy $E = \int_{-\infty}^{\infty} y^2(t)dt$ of the output $y(t) = x(t) * h(t)$.

$$y(t) = x(t) * h(t) = \frac{1}{\sigma_3\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_3^2}} \qquad \sigma_3 = 5$$

$$E = \int_{-\infty}^{\infty} y^2(t)\,dt = \frac{1}{\sigma^2 2\pi} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2(\sigma/\sqrt{2})^2}}\,dt$$

$\sigma = \sigma_3$

$$= \frac{1}{\sigma_3^2 2\pi} \sqrt{2\pi}\, \frac{\sigma_3}{\sqrt{2}} = \frac{1}{\sigma_3 2\sqrt{\pi}} = \frac{1}{10\sqrt{\pi}}$$

ANS

Problem 4 (i). The Gaussian pulse $x(t) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(t-2)^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 3^2$, is input to an LTI system with a Gaussian shaped impulse response, $h(t) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(t-3)^2}{2\sigma_2^2}}$, with $\sigma_2^2 = 4^2$. Determine the numerical value of the energy $E = \int_{-\infty}^{\infty} y^2(t)dt$ of the output $y(t) = x(t) * h(t)$.

$y_i(t) = y_f(t-5)$ from TI

$$\int_{-\infty}^{\infty} y_f^2(t-5)\,dt = \int_{-\infty}^{\infty} y_f^2(t)\,dt = \int_{-\infty}^{\infty} y_i^2(t)\,dt \Rightarrow \sigma_3 = 5$$

same answer as 4 (bi) $= \dfrac{1}{10\sqrt{\pi}}$

Problem 4 (j). The signal $z(t) = x(t)y(t)$ is the PRODUCT of the Gaussian pulse $x(t) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 3^2$, and the Gaussian pulse $y(t) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_2^2}}$, with $\sigma_2^2 = 4^2$. Determine a simple expression for the Fourier Transform, $Z(\omega)$, of $z(t) = x(t)y(t)$.

$$z(t) = x(t)\,y(t) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{t^2}{2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \sqrt{2\pi}\,\sigma_3 \frac{1}{\sqrt{2\pi}\,\sigma_3} e^{-\frac{t^2}{2\sigma_3^2}}$$

$$\sigma_3^2 = \frac{1}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$$

$$\xleftrightarrow{\mathcal{F}} \frac{1}{\sqrt{2\pi}} \frac{\sigma_3}{\sigma_1\sigma_2} e^{-\frac{\omega^2}{2\frac{1}{\sigma_3^2}}}$$