Name:

ECE301 Signals and Systems **Final Exam**

Tuesday, May 1, 2018

Cover Sheet

Write your name on this page and every page to be safe.

Test Duration: 120 minutes.

Coverage: Comprehensive

Closed Book. Four two-sided handwritten sheets.

Calculators NOT allowed.

This test contains **four** problems, each with multiple parts.

You have to draw your own plots.

You must show all work for each problem to receive full credit.

Good luck! It was great having you in class this semester!

Have a great Summer!

Problem 1.

(a) Consider the system: $y(t) = x(at)$, where a is real-valued and positive, $a > 0$. Let E_y denote the energy in $y(t) = x(at)$ and E_x denote the energy in $x(t)$. Determine the relationship between E_y and E_x in terms of a.

(b) Show that your result holds for the case where $x(t)$ is the Gaussian signal below. That is, determine E_x and E_y for the Gaussian signals below and show that they satisfy the relationship that you derived in part (a.) Your answers, of course, will also depend on the standard deviation, σ, which is a real-valued, positive constant.

$$x(t) = e^{-\frac{t^2}{2\sigma^2}} \qquad\qquad y(t) = x(at) = e^{-\frac{(at)^2}{2\sigma^2}}$$

Show your work for Problem 1 here.

Problem 2. Consider the periodic input signal $x_0(t)$ below, which is one of the entries in Table 4.2 of Fourier Transform pairs.

$$x_0(t) = \frac{2\pi}{5} \sum_{k=-\infty}^{\infty} \delta\left(t - k\frac{2\pi}{5}\right)$$

This signal is first input to an analog filter with impulse response

$$h_{LP}(t) = \frac{\sin(20t)}{\pi t} - \frac{\pi}{10}\left\{\frac{\sin(10t)}{\pi t}\right\}^2$$

to form $x(t) = x_0(t) * h_{LP}(t)$, and then $x(t)$ is sampled at a rate of $\omega_s = 40$ to form $x[n]$, so that the time between samples is $T_s = \frac{2\pi}{40}$. The DT signal $x[n]$ thus obtained is then input to a DT LTI system with impulse response

$$h_{LP}[n] = \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} - 4\left\{\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}\right\}^2 \tag{1}$$

Show all work. Write your expression for the output $y[n] = x[n] * h[n]$ in the space below. Plot both the Fourier Transform of $h_{LP}(t)$ and the DTFT of $h[n]$ to help solve the problem, but since the input is a sum of sinewaves (HINT) it is not necessary to plot the Fourier Transform of $x_0(t)$ or the DTFT of the sampled signal $x[n]$.

Problem 2. You can continue your work and plots for Problem 2 here.

Problem 2. You can continue your work and plots for Problem 2 here.

Problem 3 (a). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. That is, the Fourier Transform of the analog signal $x_a(t)$ is exactly zero for $|\omega| > 20$ rads/sec. This signal is sampled at a rate $\omega_s = 60$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{60}$ sec. This yields the discrete-time sequence

$$x[n] = x_a(nT_s) = \frac{\sin\left(\frac{2\pi}{3}n\right)}{\pi n T_s} - \frac{\pi}{10}\left\{\frac{\sin\left(\frac{\pi}{3}n\right)}{\pi n T_s}\right\}^2 \qquad \text{where: } \; T_s = \frac{2\pi}{60}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h\left(t - nT_s\right) \qquad \text{where: } \; T_s = \frac{2\pi}{60} \quad \text{and} \quad h(t) = T_s \frac{\pi}{10} \frac{\sin(10t)}{\pi t} \frac{\sin(30t)}{\pi t}$$

Problem 3 (b). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 60$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h\left(t - nT_s\right) \qquad \text{where: } \; T_s = \frac{2\pi}{60} \quad \text{and} \quad h(t) = T_s \frac{1}{2}\left\{\frac{\sin(21t)}{\pi t} + \frac{\sin(39t)}{\pi t}\right\}$$

Problem 3 (c). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 60$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h\left(t - nT_s\right) \qquad \text{where: } \; T_s = \frac{2\pi}{60} \quad \text{and} \quad h(t) = T_s \frac{\pi}{10}\left\{\frac{\sin(10t)}{\pi t}\right\}^2$$

Show all your work for Prob. 3, parts (a)-(b)-(c) on next page.

Show your work for Prob. 3, parts (a)-(b)-(c) below.

Problem 3 (d). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. That is, the Fourier Transform of the analog signal $x_a(t)$ is exactly zero for $|\omega| > 20$ rads/sec. This signal is sampled at a rate $\omega_s = 50$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{50}$ sec. This yields the discrete-time sequence

$$x[n] = x_a(nT_s) = \frac{\sin\left(\frac{4\pi}{5}n\right)}{\pi n T_s} - \frac{\pi}{10}\left\{\frac{\sin\left(\frac{2\pi}{5}n\right)}{\pi n T_s}\right\}^2 \qquad \text{where:} \quad T_s = \frac{2\pi}{50}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h\left(t - nT_s\right) \qquad \text{where:} \quad T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s\frac{1}{2}\left\{\frac{\sin(21t)}{\pi t} + \frac{\sin(29t)}{\pi t}\right\}$$

Problem 3 (e). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 50$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h\left(t - nT_s\right) \qquad \text{where:} \quad T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s\frac{\pi}{5}\left\{\frac{\sin(5t)}{\pi t}\frac{\sin(25t)}{\pi t}\right\}$$

Problem 3 (f). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is now sampled at the same rate $\omega_s = 50$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h\left(t - nT_s\right) \qquad \text{where:} \quad T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s\frac{\sin(25t)}{\pi t}$$

Show all your work for Prob. 3, parts (d)-(e)-(f) on next page.

Show your work for Prob. 3, parts (d)-(e)-(f) below.

Problem 3 (g). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at a rate $\omega_s = 30$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{30}$ sec, yielding the following discrete-time sequence:

$$x[n] = x_a(nT_s) = \frac{\sin\left(\frac{4\pi}{3}n\right)}{\pi n T_s} - \frac{\pi}{10}\left\{\frac{\sin\left(\frac{2\pi}{3}n\right)}{\pi n T_s}\right\}^2 \qquad \text{where:} \quad T_s = \frac{2\pi}{30}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a closed-form expression for the reconstructed signal $x_r(t)$. Show all work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where:} \quad T_s = \frac{2\pi}{30} \quad \text{and} \quad h(t) = T_s\frac{\sin(15t)}{\pi t}$$

Problem 3 (h). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 30$ rads/sec., where $\omega_s = 2\pi/T_s$ and the time between samples is $T_s = \frac{2\pi}{30}$ sec, but offset by $Ts/2$ from $t = 0$, i.e., at the points $t = nT_s + T_s/2$. This yields the Discrete-Time $x[n]$ signal below:

$$x[n] = x_a((n+0.5)T_s) = \frac{\sin\left(\frac{4\pi}{3}(n+0.5)\right)}{\pi(n+0.5)T_s} - \frac{\pi}{10}\left\{\frac{\sin\left(\frac{2\pi}{3}(n+0.5)\right)}{\pi(n+0.5)T_s}\right\}^2 \qquad \text{where:} \quad T_s = \frac{2\pi}{30}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x_\epsilon[n]h(t - (n+0.5)T_s) \qquad \text{where:} \quad T_s = \frac{2\pi}{30} \quad \text{and} \quad h(t) = T_s\frac{\sin(15t)}{\pi t}$$

Show your work for Prob. 3, parts (g)-(h) on the next 2 pages.

Show all your work for Prob. 3, part (g) on this page.

Show all your work for Prob. 3, part (h) on this page.

Problem 4.

The rectangular pulse $x_{in}(t) = \{u(t) - u(t-1)\}$ of duration 1 sec is input to the following system of two integrators:

$$x_a(t) = 3\int_{t-2}^{t} x_{in}(\tau)d\tau - 3\int_{t-5}^{t-3} x_{in}(\tau)d\tau$$

The output $x_a(t)$ is sampled every $T_s = 1/3$ seconds to form $x[n] = x_a(nT_s)$. The sampling rate is $f_s = 3$ samples/sec. Show work. Clearly label and write your final answer in the space provided on the next few pages.

(a) Do a stem plot of the DT signal $x[n]$ **OR** you can simply write the numbers that comprise $x[n]$ (indicate with an arrow where the $n = 0$ value is.)

(b) The Discrete-Time (DT) signal $x[n]$, created as described above, is input to the DT system described by the difference equation below:

$$y[n] = x[n] - 2x[n-1] + x[n-2]$$

(i) First, determine the impulse response $h[n]$ for this system.

(ii) Determine and do a stem-plot the output $y[n]$ by convolving the input $x[n]$ defined above with the impulse response $h[n]$ OR you can simply write $y[n]$ in sequence form. Show all work in the space provided.

Show your work and plots for Problem 4 here.

Show your work and plots for Problem 4 here.

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(\omega)$
		$y(t)$	$Y(\omega)$
4.3.0	Duality	$X(t)$	$2\pi\, x(-\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(\omega) + bY(\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
4.3.3	Conjugation	$x^*(t)$	$X^*(-\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|} X\left(\frac{\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(\omega)Y(\omega)$
4.5	Multiplication	$x(t)y(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X(\omega) * Y(\omega) =$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\theta)Y(\omega - \theta)\, d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt} x(t)$	$j\omega X(\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t) dt$	$\frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j \frac{d}{d\omega} X(\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(\omega) = X^*(-\omega) \\ \mathcal{Re}\{X(\omega)\} = \mathcal{Re}\{X(-\omega)\} \\ \mathcal{Im}\{X(\omega)\} = -\mathcal{Im}\{X(-\omega)\} \\ \|X(\omega)\| = \|X(-\omega)\| \\ \sphericalangle X(\omega) = -\sphericalangle X(-\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(\omega)$ purely imaginary and od
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{Ev}\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{Od}\{x(t)\}$ [$x(t)$ real]	$\mathcal{Re}\{X(\omega)\}$ $j\mathcal{Im}\{X(\omega)\}$

Initial Value Theorems: $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega$

$X(0) = \int_{-\infty}^{\infty} x(t)\, dt$

4.3.7 Parseval's Relation for Aperiodic Signals

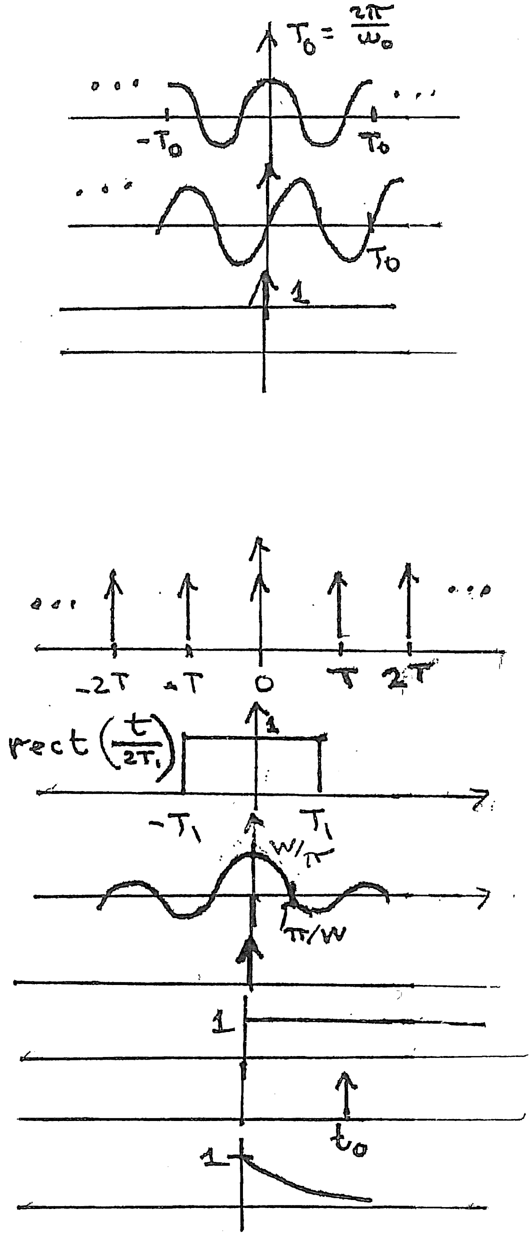
$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

4.3.8 Frequency Shift Variants

$$x(t)\cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0)$$

$$x(t)\sin(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2j} X(\omega - \omega_0) - \frac{1}{2j} X(\omega + \omega_0)$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS



Signal	Fourier transform
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$x(t) = 1$	$2\pi\,\delta(\omega)$
Periodic square wave $x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & T_1 < \|t\| \le \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$
$x(t) \begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \|\omega\| < W \\ 0, & \|\omega\| > W \end{cases}$
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$
$\frac{\pi}{W_1} \frac{\sin(W_1 t)}{\pi t} \cdot \frac{\sin(W_2 t)}{\pi t}$	
$e^{-a\|t\|}$	$\frac{2a}{a^2 + \omega^2}$
$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$	$e^{-\frac{1}{2}\omega^2\sigma^2}$
$\frac{1}{\pi t}$	$-j\,\text{sgn}(\omega) =$ j for $\omega<0$, -j for $\omega>0$