Fourier Transform Properties
Proofs and Examples

$$X(\omega) = \mathcal{F}\{x(t)\}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$x(t)$ $X(\omega)$

Time Domain

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

Frequency Domain

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$

Time-Shift Property

Inverse Fourier Transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega(t-t_0)}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{-j\omega t_0} X(\omega)\right) e^{j\omega t}\, d\omega$$

Since FT is unique:

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(\omega)$$

Modulation Property:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(\omega-\omega_0) = \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt$$

$$= \int_{-\infty}^{\infty} \left\{ e^{+j\omega_0 t} x(t) \right\} e^{-j\omega t}\, dt$$

Since FT is unique:

$$e^{j\omega_0 t} x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega-\omega_0)$$

Convolution Property: $y(t) = x(t) * h(t)$

Take FT of both sides:

$$Y(\omega) = \mathcal{F}\{y(t)\} = \mathcal{F}\left\{ \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right\}$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \mathcal{F}\{x(t-\tau)\}\, d\tau$$

time-shift property $\Rightarrow$ $$= \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau} X(\omega)\, d\tau$$

$$= X(\omega) \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$ let $\tau = t$

$$= X(\omega)\, H(\omega)$$

$$x(t) * h(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)\, H(\omega)$$

Multiplication Property

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \qquad y(t) \xleftrightarrow{\mathcal{F}} Y(\omega)$$

$$z(t) = x(t)\,y(t) \xleftrightarrow{\mathcal{F}} Z_1(\omega) = ?$$

$$Z(\omega) = \int_{-\infty}^{\infty} \{x(t)\,y(t)\} e^{-j\omega t}\,dt$$

Substitute inverse FT for $y(t)$ BUT need to use alternative dummy variable of integration rather than ω

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega) e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\lambda) e^{j\lambda t}\,d\lambda$$

$$Z(\omega) = \int_{-\infty}^{\infty} x(t) \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\lambda) e^{j\lambda t} e^{-j\omega t}\,d\lambda\,dt$$

Switch order of integration

$$Z(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\lambda) \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\lambda)t}\,dt\,d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\lambda)\, X(\omega-\lambda)\,d\lambda$$

THUS:

$$x(t)\,y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X(\omega) * Y(\omega)$$

Differentiation Property

$$\frac{d}{dt}\, x(t) \xleftrightarrow{\mathcal{F}} ??$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Take derivative wrt t on both sides of inverse FT

$$\frac{d}{dt}\, x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, \frac{d}{dt}\left(e^{j\omega t}\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{X(\omega)\, j\omega\right\} e^{j\omega t}\, d\omega$$

Thus:

$$\frac{d\,x(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega\, X(\omega)$$

Multiplication by Time Property (Differentiation in Frequency)

$$\frac{d}{d\omega}\left\{X(\omega)\right\} = \frac{d}{d\omega}\left\{\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right\}$$

$$= \int_{-\infty}^{\infty} x(t)\, \frac{d}{d\omega}\left(e^{-j\omega t}\right) dt$$

$$= \int_{-\infty}^{\infty} x(t)\,(-jt)\, e^{-j\omega t}\, dt$$

THUS:

$$-j\,t\,x(t) \xleftrightarrow{\mathcal{F}} \frac{d}{d\omega}\, X(\omega)$$

OR: $$t\,x(t) \xleftrightarrow{\mathcal{F}} j\,\frac{d}{d\omega}\, X(\omega)$$

<u>Duality</u> <u>Property</u>: This is subtly tricky.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

t is a dummy variable of integration
change to $\lambda = t$

$$X(\omega) = \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda}\, d\lambda$$

Now, replace / substitute $\omega = t$ in eqn. above

$$X(t) = \int_{-\infty}^{\infty} x(\lambda)\, e^{-jt\lambda}\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\, x(\lambda)\, e^{-j\lambda t}\, d\lambda$$

change of variables: $\omega = -\lambda$ $\quad (\lambda = -\omega)$

$d\omega = -d\lambda$ $\quad (d\lambda = -d\omega)$

$$X(t) = \frac{1}{2\pi}\int_{\infty}^{-\infty} 2\pi\, x(-\omega)\, e^{j\omega t}\, (-d\omega)$$

now in form of inverse Fourier Transform

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{2\pi\, x(-\omega)\right\} e^{j\omega t}\, d\omega$$

Thus, if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$

then $X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$

Dual Property Pair 1:

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(\omega)$$

$$e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(\omega-\omega_0)$$

Dual Property Pair 2:

$$x(t) * y(t) \xleftrightarrow{\mathcal{F}} X(\omega) Y(\omega)$$

$$x(t)\, y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X(\omega) * Y(\omega)$$

Dual Property Pair 3:

$$\frac{d}{dt} x(t) \xleftrightarrow{\mathcal{F}} j\omega X(\omega)$$

$$t\, x(t) \xleftrightarrow{\mathcal{F}} j \frac{d}{d\omega} X(\omega)$$

Duality Property:

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

$$X(t) \xleftrightarrow{\mathcal{F}} 2\pi\, x(-\omega)$$

Time-Scaling Property:

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Parseval's Theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Time-Scaling Property

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

THUS:

$$x(at) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega a t}\, d\omega$$

change of variables: $\omega' = \omega a \Rightarrow \omega = \frac{\omega'}{a}$

$d\omega' = a\, d\omega \Rightarrow d\omega = \frac{d\omega'}{a}$

assume $a > 0$:

old limits $\omega\Big]_{-\infty}^{\infty} \Rightarrow$ new limits $\Rightarrow \omega' = a\omega\Big]_{-\infty}^{\infty}$

$$x(at) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(\frac{\omega'}{a}\right) e^{j\omega' t}\, \frac{d\omega'}{a}$$

drop primes:

$$x(at) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{a} X\left(\frac{\omega}{a}\right) e^{j\omega t}\, d\omega$$

THUS:

$$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

CAN FURTHER PROVE FOR ANY **a**:

$$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Text Example 4.1

$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega} = X(\omega)$$

$$= \frac{1}{a}\,\frac{1}{1+j\frac{\omega}{a}}$$

They applied Fourier Transform integral to get this result

Note: $|X(\omega)| = \dfrac{1}{\sqrt{a^2+\omega^2}}$ $\quad \angle X(\omega) = -\tan^{-1}\left(\dfrac{\omega}{a}\right)$

See Fig. 4.5

Text Example 4.2

$y(t) = e^{-a|t|}$ $\qquad y(t) = x(t) + x(-t)$

$x(t) = e^{-at}u(t)$

$$Y(\omega) = X(\omega) + \frac{1}{|-1|}X(-\omega)$$

$$= \frac{1}{a+j\omega} + \frac{1}{a-j\omega}$$

$$= \frac{a-j\omega + a+j\omega}{a^2+\omega^2}$$

$$= \frac{2a}{a^2+\omega^2}$$

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$$

See Fig. 4.7

Text Example 4.3

$$x(t) = \delta(t) \xleftrightarrow{\mathcal{F}} X(\omega) = 1 \;\; \forall\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega \overset{\leftarrow \text{zero}}{0}}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$ area under delta fn.

Text Example 4.4

$$x(t) = \text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} X(\omega) = \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$

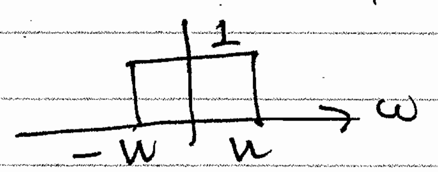
See Fig. 4.8

Text Example 4.5

See Figs. 4.9, 4.10, 4.11

$$x(t) = ? \xleftrightarrow{\mathcal{F}} X(\omega) = \text{rect}\left(\frac{\omega}{2W}\right)$$

Duality applied to eq. 4.4



$$\frac{\sin\left(T\frac{t}{2}\right)}{\frac{t}{2}} \xleftrightarrow{\mathcal{F}} 2\pi\, \text{rect}\left(\frac{-\omega}{T}\right)$$

Set $T = 2W$ and divide by 2π

$$\frac{\sin(Wt)}{\pi t} \xleftrightarrow{\mathcal{F}} \text{rect}\left(\frac{\omega}{2W}\right)$$

Example 4.6

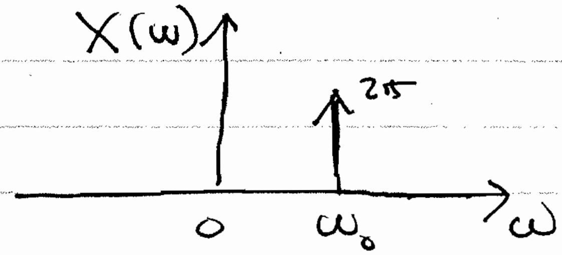
$$x(t) = e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} X(\omega) = ?$$

$$-\infty < t < \infty$$

$|X(\omega)|^2$ shows how energy of $x(t)$ is distributed as a function of frequency.

There is only <u>one</u> frequency in $x(t) = e^{j\omega_0 t}$

So guess: $X(\omega) = 2\pi\,\delta(\omega - \omega_0)$



Take inverse Fourier Transform to see if guess is correct.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(2\pi\,\delta(\omega - \omega_0)\right) e^{j\omega t}\, d\omega$$

Use sifting property of Dirac Delta function

$$x(t) = \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} \cancel{2\pi}\,\delta(\omega - \omega_0)\, e^{j\omega_0 t}\, d\omega$$

$$= e^{j\omega_0 t}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\, d\omega$$

$$= e^{j\omega_0 t}$$ CORRECT!

$$e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\,\delta(\omega - \omega_0)$$

Example 4.6 (cont.)

If $x(t) = x(t+T)$ $\forall t$, then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = ?$$

FT is a linear operator:
$$= \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta\left(\omega - k\frac{2\pi}{T}\right)$$

See Fig. 4.12

Example 4.8 ⇒ Very important for sampling theory in Chapter 7

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$ } train of Delta functions equi-spaced by T

Fourier Series
$$= \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T}}_{\substack{a_k = \frac{1}{T} \\ \forall k}} e^{jk\frac{2\pi}{T}t} \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

THUS:

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

See Fig. 4.14