

ECE301 Signals and Systems
Spring 2005
Exam III
Solutions

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1. $y[n] = x[n] - x[n-4]$

(a) Taking DTFT of the difference equation, we get

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{j\omega}) - X(e^{j\omega}) e^{-j\omega 4} \\ &= X(e^{j\omega}) [1 - e^{-j\omega 4}] \end{aligned}$$

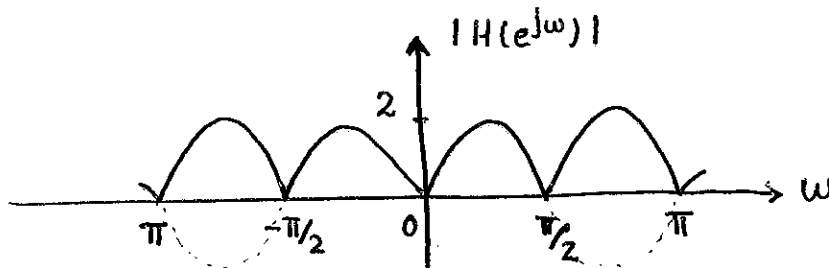
$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 - e^{-j\omega 4}$$

$$\therefore H(e^{j\omega}) = 1 - e^{-j\omega 4}$$

$$= e^{-j\omega 2} [e^{j\omega 2} - e^{-j\omega 2}]$$

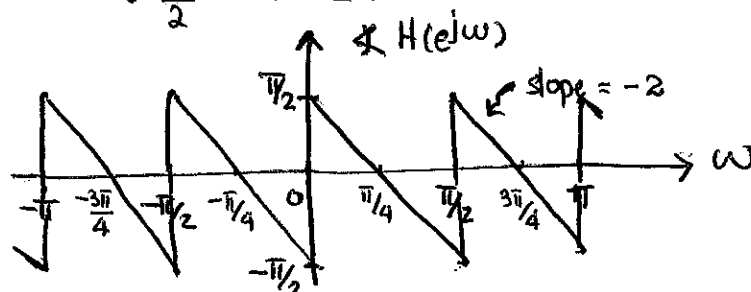
$$= 2j e^{-j\omega 2} \sin(2\omega)$$

$$|H(e^{j\omega})| = |2j e^{-j\omega 2} \sin(2\omega)| = 2 |\sin(2\omega)|$$



$$\angle H(e^{j\omega}) = \angle j + \angle e^{-j\omega 2} + \angle 2 + \angle \sin 2\omega.$$

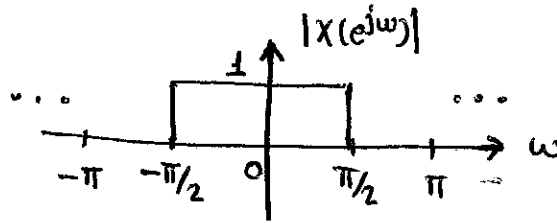
$$= \begin{cases} \frac{\pi}{2} - 2\omega & \sin(2\omega) \geq 0 \\ \frac{\pi}{2} - 2\omega \pm \pi & \sin(2\omega) < 0 \end{cases}$$



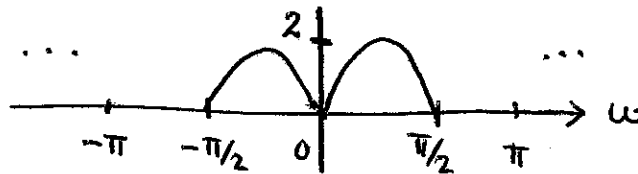
$$(b) \quad x[n] = x_a(nT_s) \quad x_a(t) = T_s \left\{ \frac{\sin(10t)}{\pi t} \right\} \quad T_s = \frac{2\pi}{40}$$

$$(i) \quad x[n] = T_s \left\{ \frac{\sin(10n \frac{2\pi}{40})}{\pi n T_s} \right\} = \frac{\sin \frac{n\pi}{2}}{\pi n}$$

From Table, $X(e^{j\omega}) = \begin{cases} 1 & 0 \leq |\omega| \leq \pi/2 \\ 0 & \pi/2 < |\omega| \leq \pi \end{cases}$



$$(ii) \quad |Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})|$$



$$(iii) \quad \sum_{n=-\infty}^{\infty} y^2[n] = ?$$

By Parseval's Relation

$$\sum_{n=-\infty}^{\infty} y^2[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Y(e^{j\omega})|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} |Y(e^{j\omega})|^2 d\omega$$

$$= \frac{1}{2\pi} 2 \int_0^{\pi/2} 2 (\sin(2\omega))^2 d\omega$$

($|Y(e^{j\omega})|$ is 0 for $\pi/2 < |\omega| < \pi$)
($|Y(e^{j\omega})|$ is even)

$$= \frac{4}{\pi} \int_0^{\pi/2} \sin^2(2\omega) d\omega$$

$$= \frac{4}{\pi} \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2} \cos(4\omega) \right) d\omega$$

$$= \frac{4}{\pi} \left[\frac{\omega}{2} - \frac{1}{2} \frac{\sin(4\omega)}{4} \right]_0^{\pi/2}$$

$$= \frac{4}{\pi} \left[\left(\frac{\pi}{4} - 0 \right) - (0 - 0) \right] = \frac{4}{\pi} \cdot \frac{\pi}{4} = 1 \leftarrow$$

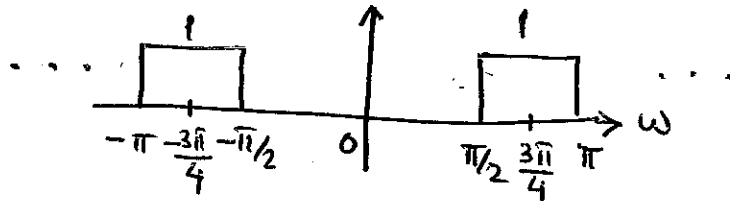
$$(c) x[n] = x_a(nT_s), \quad T_s = \frac{2\pi}{40}, \quad x_a(t) = 2T_s \left\{ \frac{\sin(st)}{\pi t} \right\} \cos(15t)$$

$$(i) x[n] = 2T_s \left\{ \frac{\sin\left(5n \frac{2\pi}{40}\right)}{\pi n T_s} \right\} \cos\left(15n \cdot \frac{2\pi}{40}\right)$$

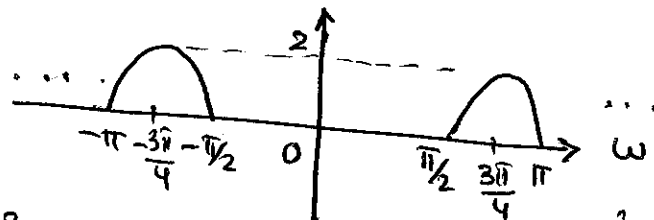
$$= 2 \left(\frac{\sin \frac{\pi n}{4}}{\pi n} \right) \cos\left(\frac{3\pi n}{4}\right)$$

$$X(e^{j\omega}) = \frac{1}{2\pi} \cdot \dots$$

$$\therefore |X(e^{j\omega})| \Rightarrow |X(e^{j\omega})|$$



$$(ii) |Y(e^{j\omega})| = |X(e^{j\omega})| |H(e^{j\omega})|$$



$$(iii) \sum_{n=-\infty}^{\infty} y^2[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Y(e^{j\omega})|^2 d\omega = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 2^2 (\sin(2\omega))^2 d\omega$$

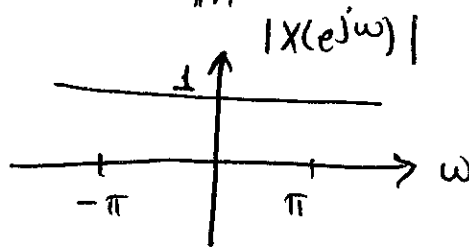
$$= \frac{4}{\pi} \left[\frac{\omega}{2} - \frac{\sin(4\omega)}{8} \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{4}{\pi} \left[\frac{\pi}{2} - 0 - \left(-\frac{\pi}{2} - 0 \right) \right] = 1 \leftarrow$$

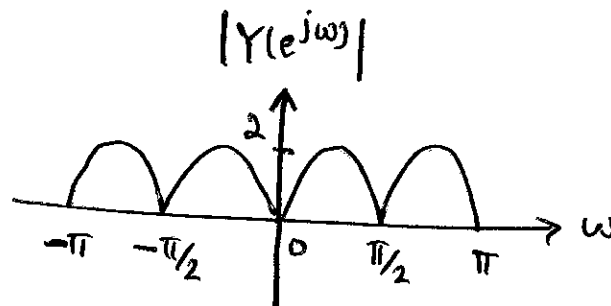
$$(d) \quad x[n] = x_a(nT_s) \quad , \quad T_s = \frac{2\pi}{20} \rightarrow x(t) = T_s \left\{ \frac{\sin(10t)}{\pi t} \right\}$$

$$(i) \quad x[n] = \frac{1}{T_s} \left\{ \frac{\sin\left(10n \frac{2\pi}{20}\right)}{\pi n \frac{1}{T_s}} \right\}$$

$$= \frac{\sin \pi n}{\pi n} = \delta[n]$$



$$(ii) \quad |Y(e^{j\omega})| = |X(e^{j\omega})| |H(e^{j\omega})| = |H(e^{j\omega})|$$



$$(iii) \quad \sum_{n=-\infty}^{\infty} y^2[n] = \frac{1}{2\pi} 4 \int_0^{\pi/2} (\sin(2\omega))^2 d\omega$$

$$= \frac{4}{2} = 2 \leftarrow$$

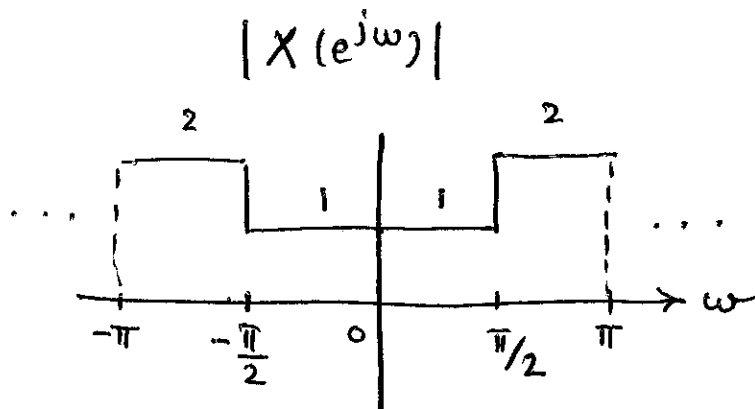
$$(e) \quad x[n] = x_a(nT_s) \quad , \quad T_s = \frac{3\pi}{20} \quad , \quad x(t) = T_s \left\{ \frac{\sin(10t)}{\pi t} \right\}$$

$$(i) \quad x[n] = \frac{1}{T_s} \left\{ \frac{\sin\left(10n \frac{3\pi}{20}\right)}{\pi n \frac{1}{T_s}} \right\}$$

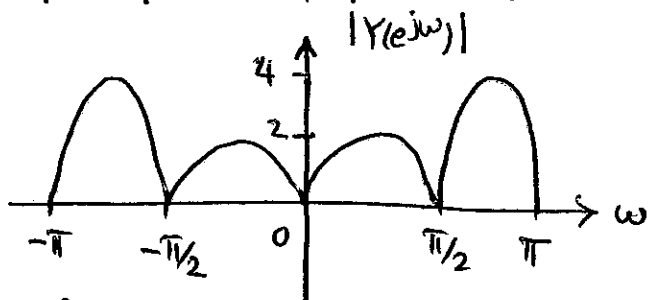
$$= \frac{\sin \frac{3\pi}{2} n}{n \frac{3\pi}{2}}$$

Nyquist Rate = 20

$$\omega_s = \frac{2\pi}{T_s} = \frac{2\pi}{3\pi/20} = \frac{20 \cdot 2}{3} = \frac{40}{3} < 20 \Rightarrow \text{Aliasing}$$



(ii) $|Y(e^{j\omega})| = |X(e^{j\omega})| |H(e^{j\omega})|$

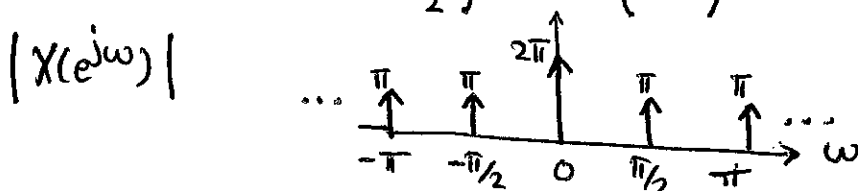


(iii) $\sum_{n=-\infty}^{\infty} y^2[n]$

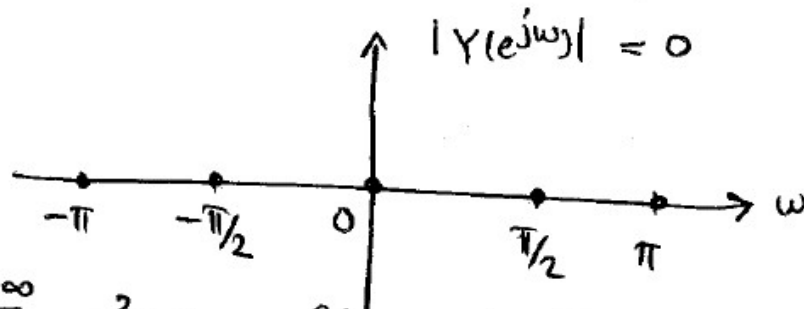
$$\begin{aligned}
 &= \frac{1}{2\pi} 2 \left[\int_0^{\pi/2} (2\sin 2\omega)^2 d\omega + \int_{\pi/2}^{\pi} (4\sin 2\omega)^2 d\omega \right] \\
 &= \frac{1}{\pi} \left[4 \int_0^{\pi/2} (\sin^2 2\omega) d\omega + 16 \int_{\pi/2}^{\pi} (\sin^2(2\omega)) d\omega \right] \\
 &= \frac{1}{\pi} \left[4 \left[\frac{\omega}{2} - \frac{1}{8} \sin(4\omega) \right]_0^{\pi/2} + 16 \left[\frac{\omega}{2} - \frac{\sin 4\omega}{8} \right]_{\pi/2}^{\pi} \right] \\
 &= \frac{1}{\pi} [\pi + 4\pi] = 5 \leftarrow
 \end{aligned}$$

(f) $x[n] = x_a(nT_s)$, $T_s = \frac{2\pi}{40}$, $x_a(t) = 1 + \cos(10t) + \cos(20t)$

(i) $x[n] = 1 + \cos\left(10n \frac{2\pi}{40}\right) + \cos\left(20n \frac{2\pi}{40}\right)$
 $= 1 + \cos\left(\frac{\pi n}{2}\right) + \cos(\pi n)$



$$(ii) |Y(e^{j\omega})| = |X(e^{j\omega})| |H(e^{j\omega})|$$

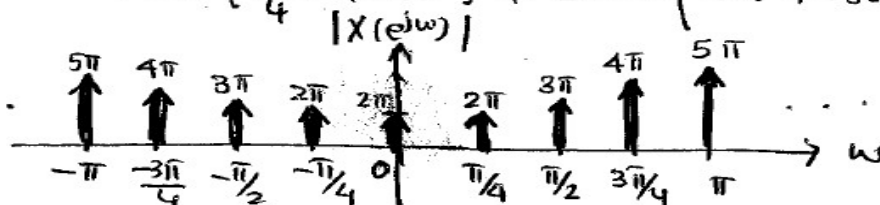


$$(iii) \sum_{n=-\infty}^{\infty} y^2[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Y(e^{j\omega})|^2 d\omega = 0 \leftarrow$$

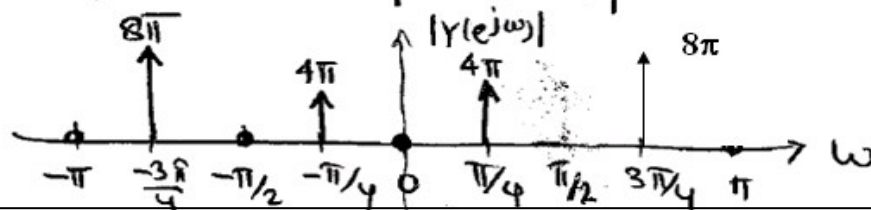
$$(g) x[n] = x_a(nT_s), \quad T_s = \frac{2\pi}{40}$$

$$(i) x_a(t) = 1 + 2 \cos(5t + 45^\circ) + 3 \cos(10t + 90^\circ) + 4 \cos(15t + 135^\circ) + 5 \cos(20t + 180^\circ)$$

$$\begin{aligned} x[n] &= 1 + 2 \cos\left(5n \frac{2\pi}{40} + 45^\circ\right) + 3 \cos\left(10n \frac{2\pi}{40} + 90^\circ\right) \\ &\quad + 4 \cos\left(15 \frac{n\pi}{40} + 135^\circ\right) + 5 \cos\left(20n \frac{2\pi}{40} + 180^\circ\right) \\ &= 1 + 2 \cos\left(\frac{\pi}{4}n + 45^\circ\right) + 3 \cos\left(\frac{\pi}{2}n + 90^\circ\right) + \\ &\quad 4 \cos\left(\frac{3\pi}{4}n + 135^\circ\right) + 5 \cos\left(\pi n + 180^\circ\right) \end{aligned}$$



$$(ii) |Y(e^{j\omega})| = |X(e^{j\omega})| |H(e^{j\omega})|$$



$$\begin{aligned}
 \text{(iii)} \quad \sum_{n=-\infty}^{\infty} y^2[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |Y(e^{j\omega})|^2 d\omega \\
 &= \frac{1}{2\pi} \cdot 2 \cdot [(4\pi)^2 + (8\pi)^2] \\
 &= \frac{2}{2\pi} [16\pi^2 + 64\pi^2] \\
 &= \frac{80\pi^2}{\pi} \\
 &= 80\pi \leftarrow
 \end{aligned}$$

$$2. \quad y[n] = x[n] + x[n-1] + x[n-2] + x[n-3]$$

$$\therefore h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$$

$$H(e^{j\omega}) = 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega}$$

$$= \sum_{n=0}^3 e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega 4}}{1 - e^{-j\omega}}$$

$$, \quad \begin{aligned} (1 - e^{-j\omega}) &\neq 0 \\ e^{-j\omega} &\neq 1 \end{aligned}$$

$$= \frac{e^{-j\omega 2} (e^{j\omega 2} - e^{-j\omega 2})}{e^{-j\omega/2} (e^{j\omega/2} - e^{-j\omega/2})}$$

$$\omega \neq 2k\pi$$

$$k=0, \pm 1, \pm 2, \dots$$

$$= e^{-j\omega 3/2} \frac{\sin(2\omega)}{\sin(\omega/2)}$$

$$\therefore |H(e^{j\omega})| = \begin{cases} \left| \frac{\sin(2\omega)}{\sin(\omega/2)} \right| & \omega \neq 0, \pm 2\pi, \pm 4\pi, \dots \\ 4 & \omega = 0 \end{cases}$$

$$|H(e^{j0})| = 1 + e^{-j0} + e^{-j0} + e^{-j0} = 1 + 1 + 1 + 1 = 4$$

$$\omega = 0,$$

$$|H(e^{j\omega})| = 4$$

$$\omega = +\pi/4,$$

$$|H(e^{j\omega})| = \left| \frac{\sin 2(\pi/4)}{\sin(\pi/4)} \right| = \left| \frac{\sin(\pi/2)}{\sin(\pi/4)} \right| = \frac{1}{\sin(\pi/4)}$$

$$\omega = -\pi/4,$$

$$|H(e^{j\omega})| = \frac{1}{\sin(\pi/4)}$$

$$\omega = +\pi/2,$$

$$|H(e^{j\omega})| = \left| \frac{\sin 2(\pi/2)}{\sin(\pi/2)} \right| = \left| \frac{\sin(\pi)}{\sin(\pi/2)} \right| = 0$$

$$\omega = -\pi/2$$

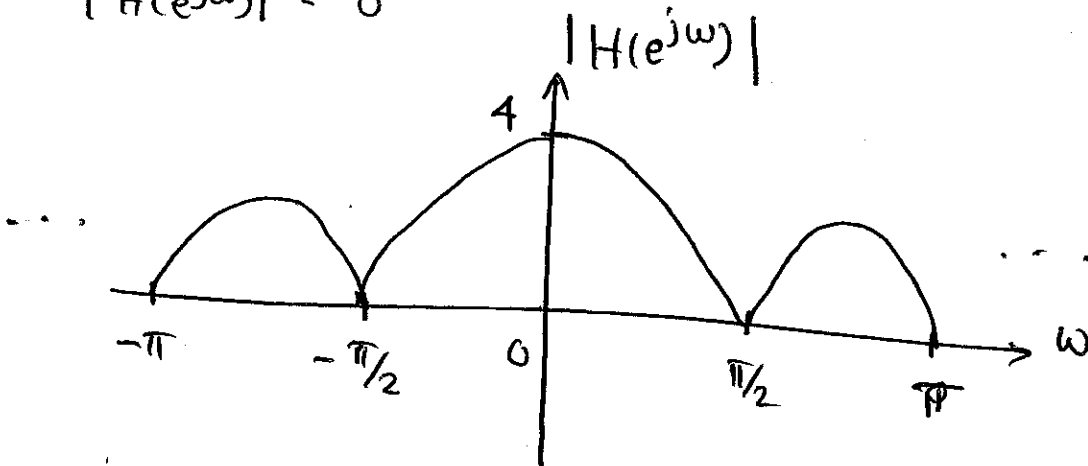
$$|H(e^{j\omega})| = 0$$

$$\omega = +\pi$$

$$|H(e^{j\omega})| = \left| \frac{\sin(2\pi)}{\sin(\pi/2)} \right| = 0$$

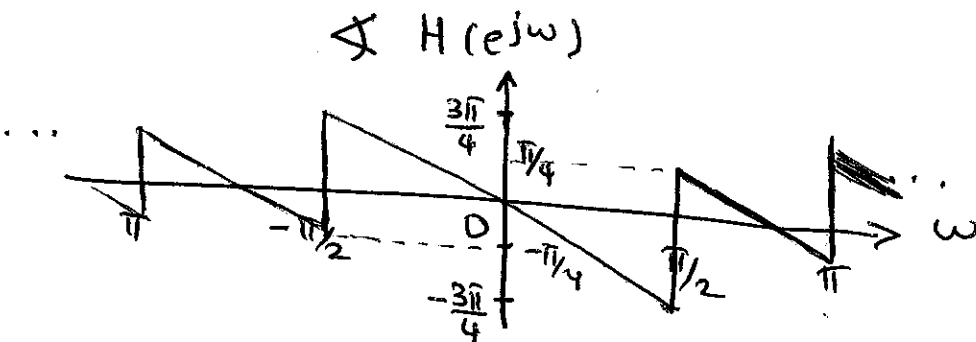
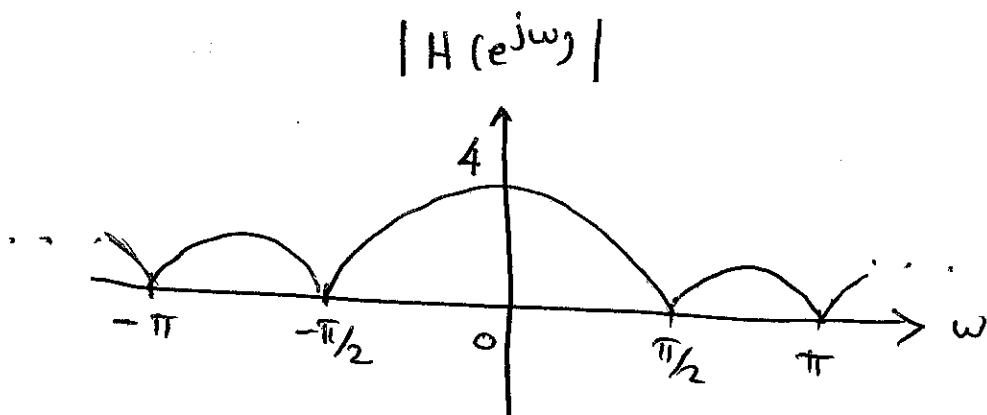
$$\omega = -\pi$$

$$|H(e^{j\omega})| = 0$$

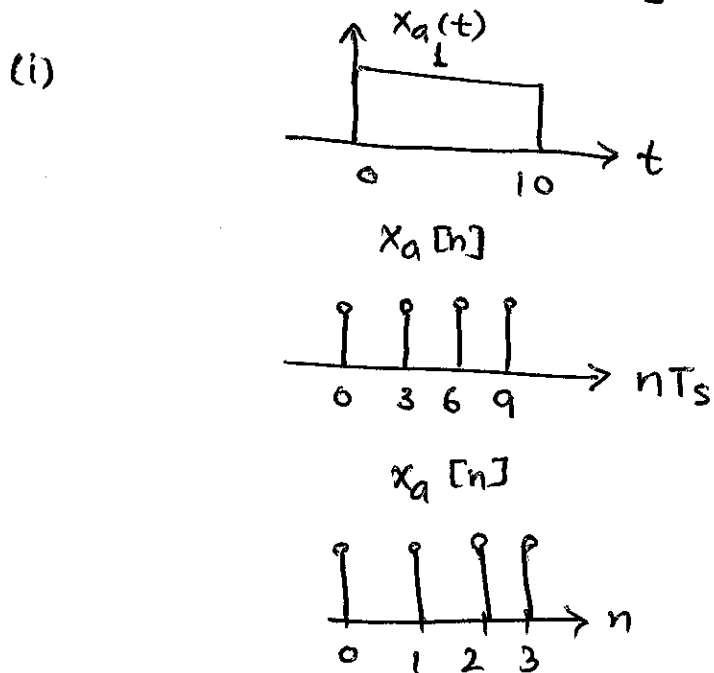


$$\angle H(e^{j\omega}) = \angle e^{-j\omega 3/2} + \angle \frac{\sin(2\omega)}{\sin(\omega/2)}$$

$$= \begin{cases} -3/2 \omega & \frac{\sin(2\omega)}{\sin(\omega/2)} \geq 0 \\ -3/2 \omega \pm \pi & \frac{\sin(2\omega)}{\sin(\omega/2)} < 0 \end{cases}$$



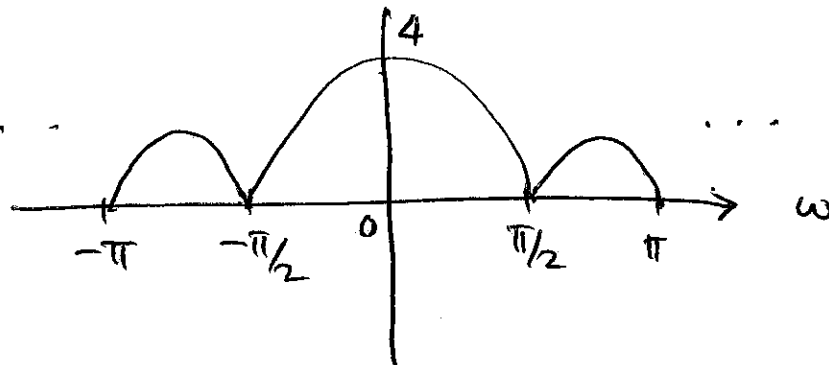
(b) $x[n] = x_a(nT_s)$ $T_s = 3$, $x_a(t) = u(t) - u(t-10)$



$x_a[n] = h[n]$

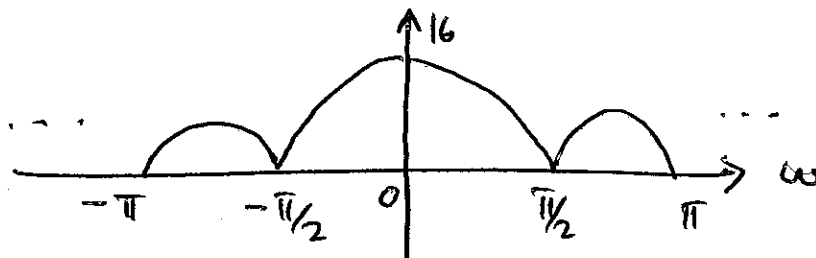
$\therefore |X(e^{j\omega})| = |H(e^{j\omega})|$

$$|X(e^{j\omega})|$$



$$\begin{aligned} \text{(ii)} \quad |Y(e^{j\omega})| &= |X(e^{j\omega})| |H(e^{j\omega})| \\ &= |H(e^{j\omega})|^2 \end{aligned}$$

$$|Y(e^{j\omega})|$$



$$\text{(iii)} \quad \sum_{n=-\infty}^{\infty} y^2[n]$$

$$y[n] = x[n] * h[n] =$$

A discrete convolution diagram showing the convolution of two sequences. The top row shows two sequences: $x[n]$ with values $0, 1, 2, 3$ at $n=0, 1, 2, 3$ and $h[n]$ with values $0, 1, 2, 3$ at $n=0, 1, 2, 3$. A vertical arrow points to the result sequence $y[n]$ with values $0, 1, 2, 3, 4, 3, 2, 1$ at $n=0, 1, 2, 3, 4, 5, 6$. The convolution is performed by shifting the sequences and summing the products of overlapping elements.

$$\begin{aligned} \sum_{n=-\infty}^{\infty} y^2[n] &= 1^2 + 2^2 + 3^2 + 4^2 + 3^2 + 2^2 + 1^2 \\ &= 1 + 4 + 9 + 16 + 9 + 4 + 1 \\ &= 44 \leftarrow \end{aligned}$$