ECE301 Signals and Systems
Spring 2005
Exam III
Solutions

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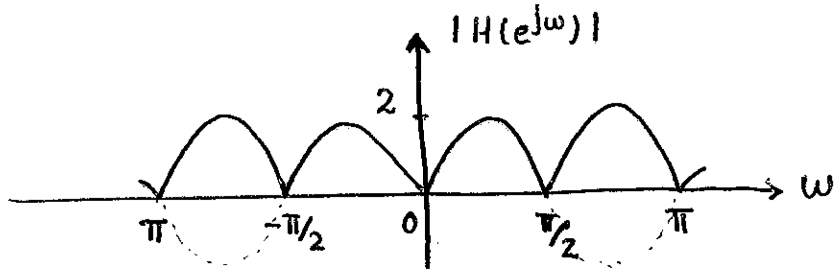
1. $y[n] = x[n] - x[n-4]$

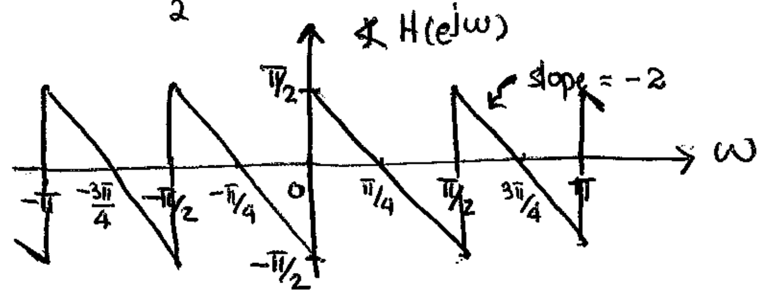
(a) Taking DTFT of the difference equation, we get

$$Y(e^{j\omega}) = X(e^{j\omega}) - X(e^{j\omega})\, e^{-j\omega 4}$$
$$= X(e^{j\omega})\,[1 - e^{-j\omega 4}]$$
$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 - e^{-j\omega 4}$$
$$\therefore \; H(e^{j\omega}) = 1 - e^{-j\omega 4}$$
$$= e^{-j\omega 2}\,[e^{j\omega 2} - e^{-j\omega 2}]$$
$$= 2j\, e^{-j\omega 2} \sin(2\omega)$$
$$|H(e^{j\omega})| = |2j e^{-j\omega 2} \sin(2\omega)| = 2\,|\sin(2\omega)|$$



$$\measuredangle H(e^{j\omega}) = \measuredangle j + \measuredangle e^{-j\omega 2} + \measuredangle 2 + \measuredangle \sin 2\omega$$

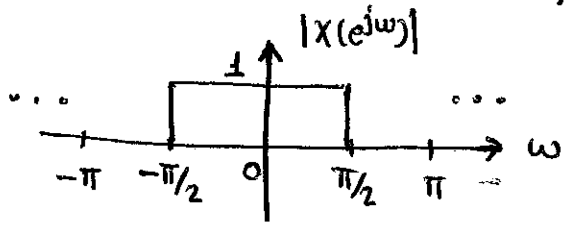
$$= \begin{cases} \frac{\pi}{2} - 2\omega & \sin(2\omega) \geq 0 \\ \frac{\pi}{2} - 2\omega \pm \pi & \sin(2\omega) < 0 \end{cases}$$



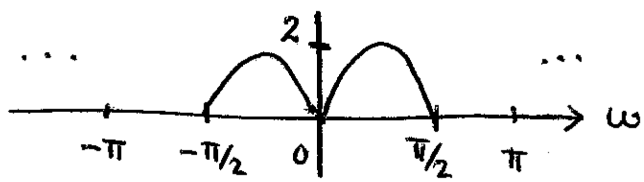
(b) $x[n] = x_a(nT_s)$ $\qquad x_a(t) = T_s \left\{ \frac{\sin(10t)}{\pi t} \right\}$ $\qquad T_s = \frac{2\pi}{40}$

(i) $x[n] = \cancel{T_s} \left\{ \frac{\sin(10\, n \frac{2\pi}{40})}{\pi n \cancel{T_s}} \right\} = \frac{\sin \frac{n\pi}{2}}{\pi n}$

From Table, $X(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| \le \pi/2 \\ 0 & \pi/2 < |\omega| \le \pi \end{cases}$



(ii)
$|Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})|$



(iii) $\sum_{n=-\infty}^{\infty} y^2[n] = ?$

By Parseval's Relation

$$\sum_{n=-\infty}^{\infty} y^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Y(e^{j\omega})|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} |Y(e^{j\omega})|^2 d\omega$$

($|Y(e^{j\omega})|$ is 0 for $\pi/2 < |\omega| < \pi$)

$$= \frac{1}{2\pi}\, 2 \int_0^{\pi/2} 2^2 (\sin(2\omega))^2 d\omega$$

($|Y(e^{j\omega})|$ is even)

$$= \frac{4}{\pi}\int_0^{\pi/2} \sin^2(2\omega)\, d\omega$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos(4\omega) \right) d\omega$$

$$= \frac{4}{\pi}\left[\frac{\omega}{2} - \frac{1}{2}\,\frac{\sin(4\omega)}{4} \right]_0^{\pi/2}$$

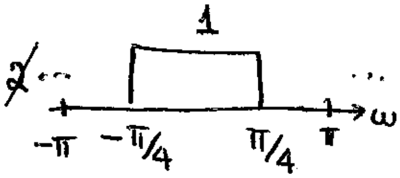
$$= \frac{4}{\pi}\left[\left(\frac{\pi}{4} - 0\right) - (0 - 0)\right] = \frac{4}{\pi}\cdot\frac{\pi}{4} = 1 \leftarrow$$

(c) $x[n] = x_a(nT_s)$, $T_s = \frac{2\pi}{40}$, $x_a(t) = 2T_s\left\{\frac{\sin(5t)}{\pi t}\right\}\cos(15t)$

(i) $x[n] = 2\cancel{T_s}\left\{\frac{\sin\left(5n\frac{2\pi}{40}\right)}{\pi n \cancel{T_s}}\right\}\cos\left(15n\cdot\frac{2\pi}{40}\right)$

$$= 2\left(\frac{\sin\frac{\pi}{4}n}{\pi n}\right)\cos\left(\frac{3\pi}{4}n\right)$$

$X(e^{j\omega}) = \frac{1}{\cancel{2\pi}}\cdot\cancel{2}$



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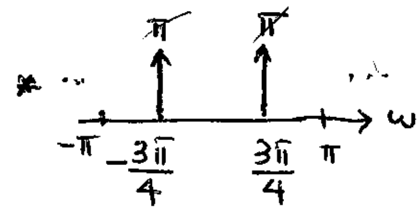

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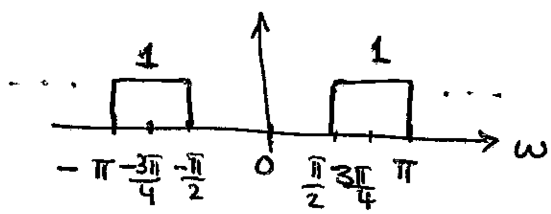
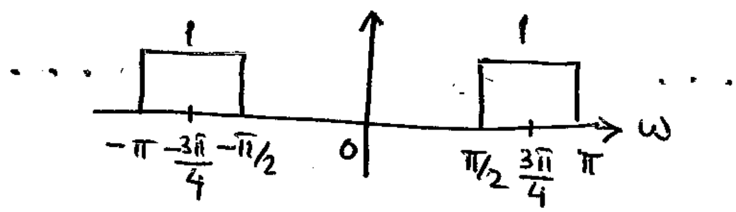
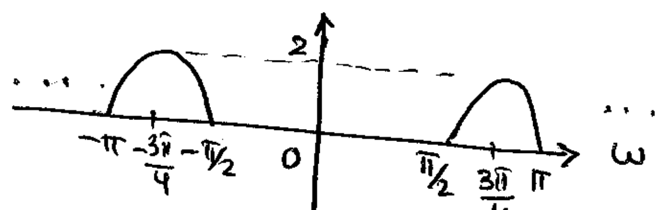


$\therefore$ $|X(e^{j\omega})| \Rightarrow |X(e^{j\omega})|$



(ii) $|Y(e^{j\omega})| = |X(e^{j\omega})|\,|H(e^{j\omega})|$



(iii) $\sum_{n=-\infty}^{\infty} y^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}|Y(e^{j\omega})|^2\,d\omega = \frac{1}{2\pi}2\int_{\pi/2}^{\pi} 2^2(\sin(2\omega))^2\,d\omega$

$$= \frac{4}{\pi}\left[\frac{\omega}{2} - \frac{\sin(4\omega)}{8}\right]_{\pi/2}^{\pi}$$

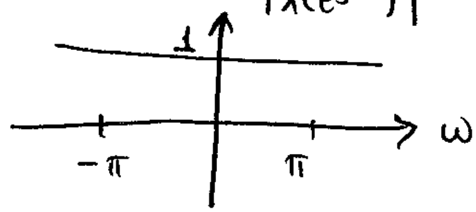
$$= \frac{4}{\pi}\left[\frac{\pi}{2} - 0 - \frac{\pi}{4} - 0\right] = 1 \leftarrow$$

(d) $x[n] = x_a(nT_s)$, $T_s = \frac{2\pi}{20} \rightarrow x(t) = T_s\left\{\frac{\sin(10t)}{\pi t}\right\}$

(i) $x[n] = \cancel{T_s}\left\{\frac{\sin\left(10n\frac{2\pi}{20}\right)}{\pi n \cancel{T_s}}\right\}$

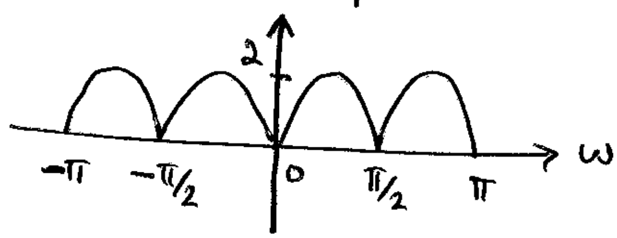
$= \frac{\sin \pi n}{\pi n} = \delta[n]$

$|X(e^{j\omega})|$



(ii) $|Y(e^{j\omega})| = |X(e^{j\omega})||H(e^{j\omega})| = |H(e^{j\omega})|$

$|Y(e^{j\omega})|$



(iii) $\sum_{n=-\infty}^{\infty} y^2[n] = \frac{1}{2\pi}\, 4 \int_0^{\pi/2} (2\sin(2\omega))^2 d\omega$

$= \frac{4}{2} = 2 \leftarrow$

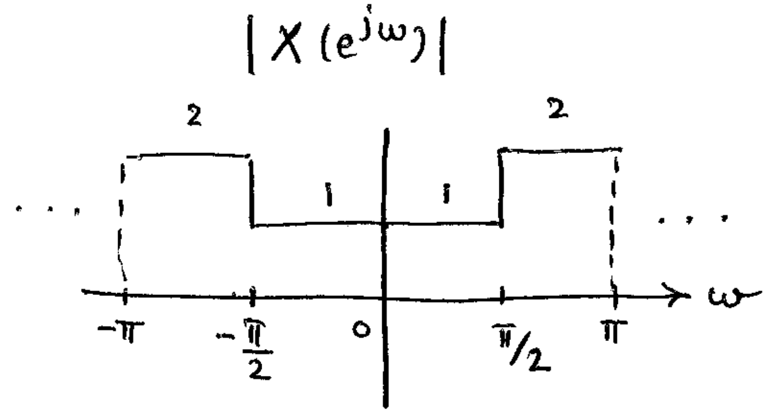
(e) $x[n] = x_a(nT_s)$, $T_s = \frac{3\pi}{20}$, $x(t) = T_s\left\{\frac{\sin(10t)}{\pi t}\right\}$

(i) $x[n] = \cancel{T_s}\left\{\frac{\sin\left(10n\frac{3\pi}{20}\right)}{\pi n \cancel{T_s}}\right\}$

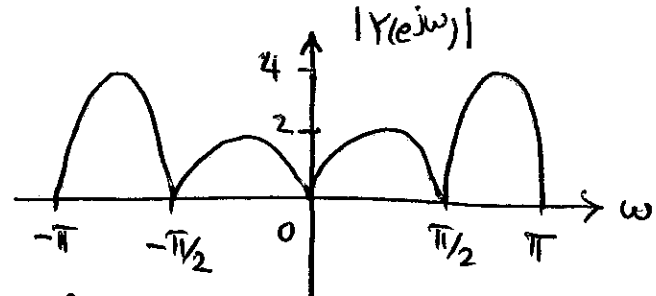
$= \frac{\sin \frac{3\pi}{2} n}{n\pi}$

Nyquist Rate $= 20$

$\omega_s = \frac{2\pi}{T_s} = \frac{2\pi}{3\pi/20} = \frac{20}{3}\cdot 2 = \frac{40}{3} < 20 \Rightarrow$ Aliasing



(ii) $|Y(e^{j\omega})| = |X(e^{j\omega})|\,|H(e^{j\omega})|$



(iii) $\sum_{n=-\infty}^{\infty} y^2[n]$

$$= \frac{1}{2\pi} 2\left[\int_0^{\pi/2} (2\sin 2\omega)^2 d\omega + \int_{\pi/2}^{\pi} (4\sin 2\omega)^2 d\omega\right]$$

$$= \frac{1}{\pi}\left[4\int_0^{\pi/2} (\sin^2 2\omega)\, d\omega + 16\int_{\pi/2}^{\pi} (\sin^2(2\omega))d\omega\right]$$

$$= \frac{1}{\pi}\left[4\left[\frac{\omega}{2} - \frac{1}{8}\sin(4\omega)\right]_0^{\pi/2} + 16\left[\frac{\omega}{2} - \frac{\sin 4\omega}{8}\right]_{\pi/2}^{\pi}\right]$$

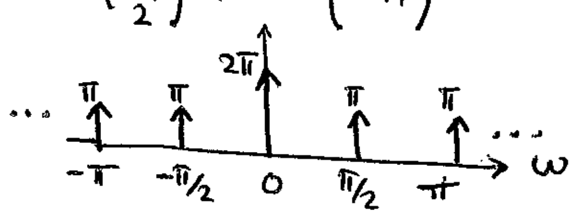
$$= \frac{1}{\pi}\left[\pi + 4\pi\right] = 5 \longleftarrow$$

(f) $x[n] = x_a(nT_s)$, $T_s = \frac{2\pi}{40}$, $x_a(t) = 1 + \cos(10t) + \cos(20t)$

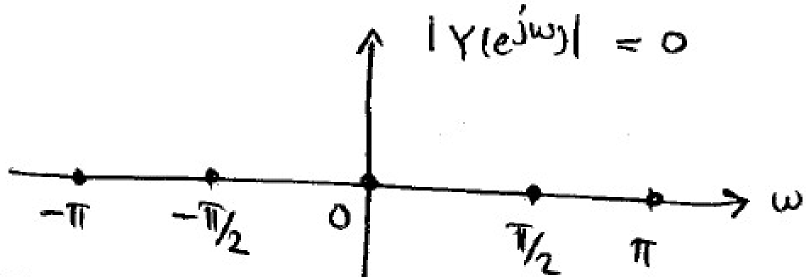
(i) $x[n] = 1 + \cos\left(10\, n \frac{2\pi}{40}\right) + \cos\left(20\, n \frac{2\pi}{40}\right)$

$$= 1 + \cos\left(\frac{\pi}{2}n\right) + \cos(\pi n)$$

$|X(e^{j\omega})|$



(ii) $|Y(e^{j\omega})| = |X(e^{j\omega})|\,|H(e^{j\omega})|$



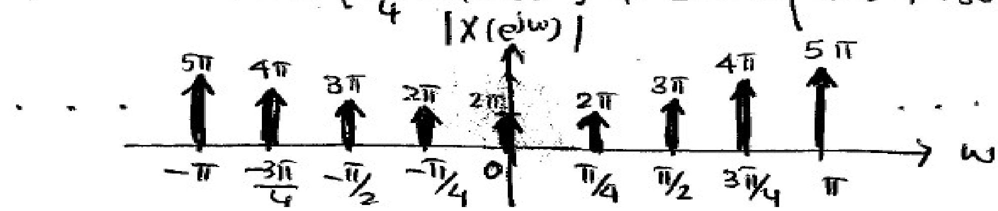
(iii) $\sum_{n=-\infty}^{\infty} y^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Y(e^{j\omega})|^2 d\omega = 0 \leftarrow$

(g) $x[n] = x_a(nT_s)$, $T_s = \frac{2\pi}{40}$

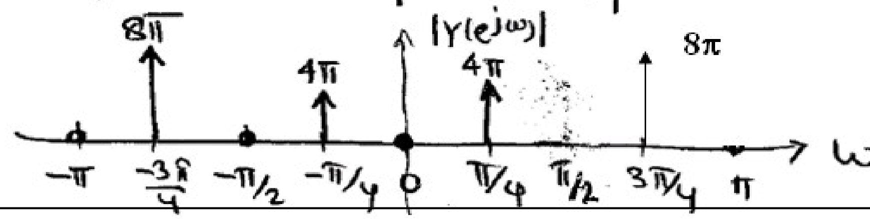
(i) $x_a(t) = 1 + 2\cos(5t + 45^\circ) + 3\cos(10t + 90^\circ) + 4\cos(15t + 135^\circ) + 5\cos(20t + 180^\circ)$

$$x[n] = 1 + 2\cos\left(5n\frac{2\pi}{40} + 45^\circ\right) + 3\cos\left(10n\frac{2\pi}{40} + 90^\circ\right) + 4\cos\left(15\frac{n\pi 2}{40} + 135^\circ\right) + 5\cos\left(20n\frac{2\pi}{40} + 180^\circ\right)$$

$$= 1 + 2\cos\left(\frac{\pi}{4}n + 45^\circ\right) + 3\cos\left(\frac{\pi}{2}n + 90^\circ\right) + 4\cos\left(\frac{3\pi}{4}n + 135^\circ\right) + 5\cos\left(\pi n + 180^\circ\right)$$



(ii) $|Y(e^{j\omega})| = |X(e^{j\omega})|\,|H(e^{j\omega})|$



(iii) $$\sum_{n=-\infty}^{\infty} y^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Y(e^{j\omega})|^2 d\omega$$

$$= \frac{1}{2\pi}\, 2\, \left[(4\pi)^2 + (8\pi)^2\right]$$

$$= \frac{2}{2\pi}\left[16\pi^2 + 64\pi^2\right]$$

$$= \frac{80\pi^2}{\pi}$$

$$= 80\pi \leftarrow$$

2. $y[n] = x[n] + x[n-1] + x[n-2] + x[n-3]$

$\therefore\ h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$

$$H(e^{j\omega}) = 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega}$$

$$= \sum_{n=0}^{3} e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega 4}}{1 - e^{-j\omega}}, \quad (1 - e^{-j\omega}) \neq 0$$

$$e^{-j\omega} \neq 1$$

$$= \frac{e^{-j\omega 2}\left(e^{j\omega 2} - e^{-j\omega 2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} \qquad \omega \neq 2k\pi$$

$$k = 0, \pm 1, \pm 2, \ldots$$

$$= e^{-j\omega 3/2}\, \frac{\sin(2\omega)}{\sin\left(\frac{\omega}{2}\right)}$$

$$\therefore\ |H(e^{j\omega})| = \begin{cases} \left|\dfrac{\sin(2\omega)}{\sin\left(\frac{\omega}{2}\right)}\right| & \omega \neq 0, \pm 2\pi, \pm 4\pi, \ldots \\ 4 & \omega = 0 \end{cases}$$

$$\boxed{H(e^{j0}) = 1 + e^{-j0} + e^{-j0} + e^{-j0} = 1 + 1 + 1 + 1 = 4}$$

$\omega = 0$,

$$|H(e^{j\omega})| = 4$$

$\omega = +\pi/4$,

$$|H(e^{j\omega})| = \left| \frac{\sin 2(\frac{\pi}{4})}{\sin(\frac{\pi/4}{2})} \right| = \left| \frac{\sin(\frac{\pi}{2})}{\sin(\frac{\pi}{8})} \right| = \frac{1}{\sin(\frac{\pi}{8})}$$

$\omega = -\frac{\pi}{4}$,

$$|H(e^{j\omega})| = \frac{1}{\sin(\frac{\pi}{8})}$$

$\omega = +\pi/2$,

$$|H(e^{j\omega})| = \left| \frac{\sin 2(\frac{\pi}{2})}{\sin(\frac{1}{2}(\pi/2))} \right| = \left| \frac{\sin(\pi)}{\sin(\pi/2)} \right| = 0$$

$\omega = -\pi/2$

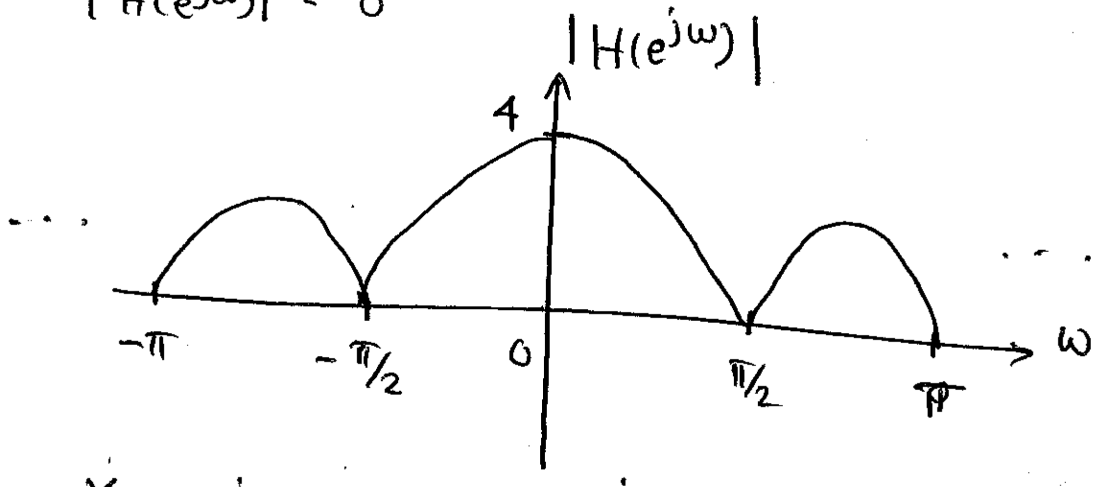
$$|H(e^{j\omega})| = 0$$

$\omega = +\pi$

$$|H(e^{j\omega})| = \left| \frac{\sin(2\pi)}{\sin(\frac{\pi}{2})} \right| = 0$$

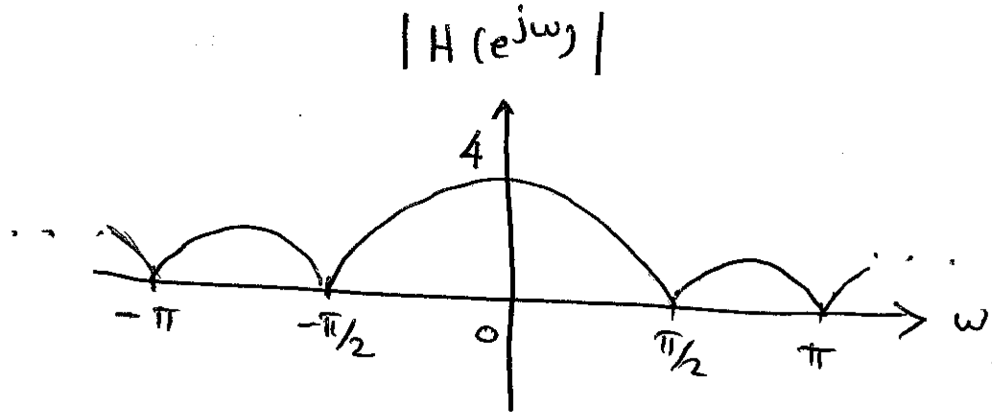
$\omega = -\pi$

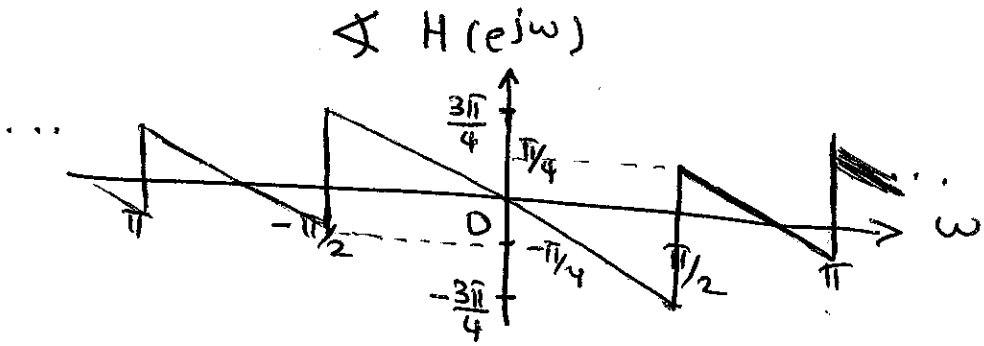
$$|H(e^{j\omega})| = 0$$



$$\measuredangle H(e^{j\omega}) = \measuredangle e^{-j\omega 3/2} + \measuredangle \frac{\sin(2\omega)}{\sin(\frac{\omega}{2})}$$

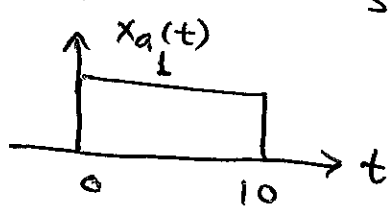
$$= \begin{cases} -\frac{3}{2}\omega & \frac{\sin(2\omega)}{\sin(\frac{\omega}{2})} \geq 0 \\ -\frac{3}{2}\omega \pm \pi & \frac{\sin(2\omega)}{\sin(\omega/2)} < 0 \end{cases}$$

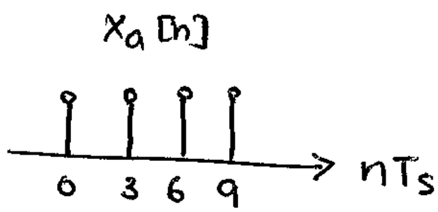


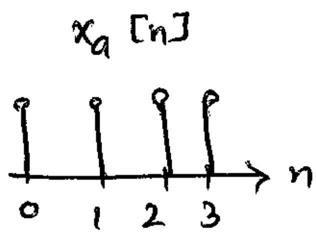


(b) $x[n] = x_a(nT_s)$ $T_s = 3$, $x_a(t) = u(t) - u(t-10)$

(i)



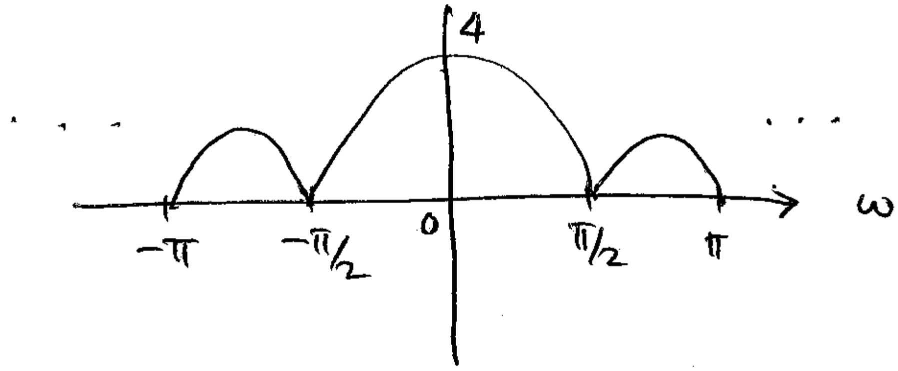




$x_a[n] = h[n]$

$\therefore |X(e^{j\omega})| = |H(e^{j\omega})|$

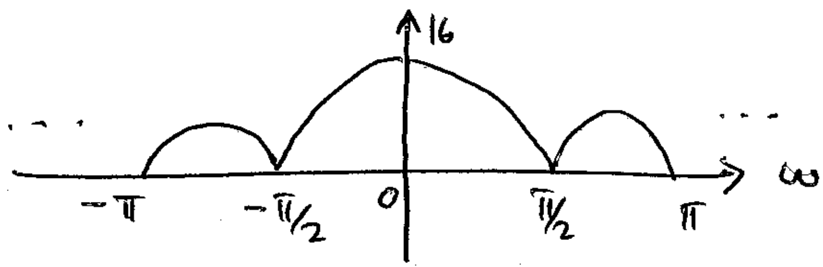
$$|X(e^{j\omega})|$$



(ii) $|Y(e^{j\omega})| = |X(e^{j\omega})|\,|H(e^{j\omega})|$

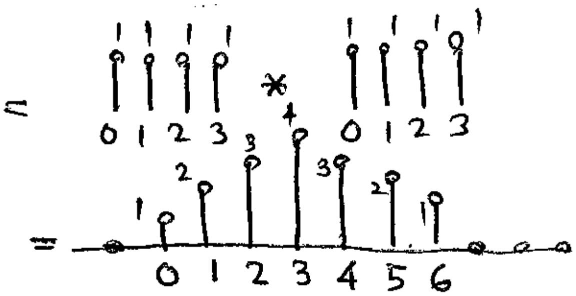
$$= |H(e^{j\omega})|^2$$

$$|Y(e^{j\omega})|$$



(iii) $\displaystyle\sum_{n=-\infty}^{\infty} y^2[n]$

$y[n] = x[n] * h[n] =$



$$\sum_{n=-\infty}^{\infty} y^2[n] = 1^2 + 2^2 + 3^2 + 4^2 + 3^2 + 2^2 + 1^2$$

$$= 1 + 4 + 9 + 16 + 9 + 4 + 1$$

$$= 44 \leftarrow$$