SOLUTION

EE301 Signals and Systems — In-Class Exam
Exam 3 — Thursday, Apr. 19, 2012

Cover Sheet

Test Duration: 75 minutes.
Coverage: Chaps. 5,7
Open Book but Closed Notes.
One 8.5 in. x 11 in. crib sheet
Calculators NOT allowed.
All work should be done on the sheets provided.
You must show all work for each problem to receive full credit.
Plot your answers on the graphs provided.

VIP Note Regarding DTFT Plots: *The abscissa in each plot is the frequency axis. For each plot, the abscissa goes from* $-\pi$ *to* π *with tic marks every* $\pi/16$. *There is a dashed vertical line at* $\omega = -\frac{\pi}{2}$ *and another dashed vertical line at* $\omega = +\frac{\pi}{2}$. *You only have to plot any DTFT over* $-\pi < \omega < \pi$.

True False Questions. Circle the True (T) or False (F) for each part below.

(T) (F) The time-domain variable for the DTFT is a discrete (integer-valued) variable for the DTFT whereas the time-domain variable for the CTFT is a continuous-valued variable.

(T) (F) The DTFT is always periodic with period 2π whereas the CTFT is generally not periodic except under special conditions.

(T) (F) The DTFT is a summation over time whereas the CTFT is an integration over time.

(T) (F) The inverse DTFT is a summation over frequency whereas the inverse CTFT is an integration over frequency.

(T) (F) The highest frequency for the DTFT is effectively π, corresponding to half of the sampling rate, whereas the highest frequency for the CTFT is ∞ .

(T) (F) When you pass an analog signal through an LTI filter, the Nyquist rate for the output signal is greater than the Nyquist rate for the input signal.

(T) (F) When you form the product of two analog signals, the Nyquist rate for the product signal is the sum of the respective Nyquist rates for the two individual signals.

(T) (F) The Nyquist rate for the square of a signal is twice the Nyquist rate of the original signal.

(T) (F) The Nyquist rate for the derivative of a signal is greater than the Nyquist rate for the original signal.

(T) (F) When you multiply a signal by a real-valued sinewave, the Nyquist rate of the resulting modulated signal is greater than the Nyquist rate for the original signal.

Problem 2 (a). Consider a CT signal $x_a(t)$ with bandwidth (maximum frequency) W in rads/sec. The sampling rate is chosen to be above the Nyquist rate at $\omega_s = 4W$, where $\omega_s = 2\pi/T_s$. $x_a(t)$ is reconstructed perfectly according to the formula below. Let $H(\omega)$ be the CTFT of $h(t)$. Determine the respective values of ω_1 and ω_2, both in terms of W, so that the CTFT $H(\omega)$ is flat up to the bandwidth W and then rolls off to zero at $\omega_s - W$.

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)h(t - nT_s) \quad \text{where} \quad h(t) = T_s\frac{\pi}{\omega_1}\frac{\sin(\omega_1 t)}{\pi t}\frac{\sin(\omega_2 t)}{\pi t} \quad \text{and} \quad \omega_s = 4W$$

$$\omega_2 - \omega_1 = W$$

$$\omega_2 + \omega_1 = \omega_s - W = 4W - W = 3W$$

$$2\omega_2 = 4W \Rightarrow \omega_2 = 2W$$

$$\omega_1 = \omega_2 - W \Rightarrow \omega_1 = W$$

Problem 2 (b). Determine an expression for the DTFT $X(\omega)$ in terms of α and N for the finite-length geometric sequence $x[n]$ below. Be sure to indicate which DTFT properties and/or pairs you use to arrive at your answer.

$$x[n] = \alpha^n\{u[n] - u[n-N]\}$$

$$x[n] = \alpha^n u[n] - \alpha^N \alpha^{n-N} u[n-N]$$

$$X(\omega) = \frac{1}{1-\alpha e^{-j\omega}} - \alpha^N \frac{e^{-j\omega N}}{1-\alpha e^{-j\omega}}$$

$$= \frac{1-\alpha^N e^{-j\omega N}}{1-\alpha e^{-j\omega}}$$

Problem 2 (c). Consider the signal $x_p(t)$ below, which is the Fourier Series expansion for a periodic sawtooth waveform with period $T = 1$ sec.

$$x_p(t) = \sum_{k=-\infty}^{-1} \frac{j(-1)^k}{k\pi} e^{j2\pi kt} + \sum_{k=1}^{\infty} \frac{j(-1)^k}{k\pi} e^{j2\pi kt}$$

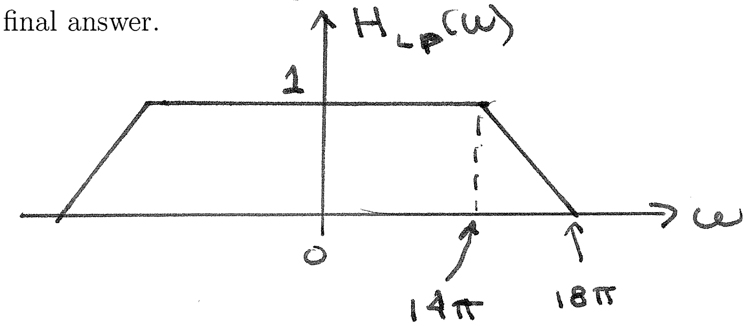
This signal is first low-passed filtered with an analog lowpass filter with impulse response

$$h_{LP}(t) = \frac{1}{2} \frac{\sin(2\pi 8t)}{\pi t} \frac{\sin(2\pi t)}{\pi t}$$

to form $x(t) = x_p(t) * h_{LP}(t)$ and then $x(t)$ is sampled at a rate of $F_s = \frac{1}{T_s} = 16$ samples/sec to form $x[n]$, so that the time between samples is $T_s = \frac{1}{16}$. (In terms of radians, the sampling rate is: $\omega_s = 2\pi F_s = \frac{2\pi}{T_s} = 2\pi 16$.) The DT signal $x[n]$ thus obtained is then the input to a DT LTI system with impulse response

$$h[n] = 8 \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} \frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n} \tag{1}$$

Show all work. Write your expression for the output $y[n] = x[n] * h[n]$ in the space below. Plot the DTFT of $h[n]$ to help determine the gain of each sinewave gets as it passes thru the system. Be sure to clear about the output amplitude and frequency of each sine wave at the output in your final answer.



Thus, the frequencies that pass with unity gain are $k\,2\pi$, $k = -7, \ldots, -1, 1, \ldots, 7$

Since 16π is half way between 14π and 18π, it gets a gain of $1/2$ as it passes thru the filter

$$x(t) = \frac{1}{2} \frac{j(-1)^{-8}}{8\pi} e^{-j16\pi t} + \sum_{\substack{k=-7 \\ k\neq 0}}^{7} \frac{j(-1)^k}{k\pi} e^{j2\pi kt} + \frac{1}{2} \frac{j(-1)^8}{8\pi} e^{j16\pi t}$$

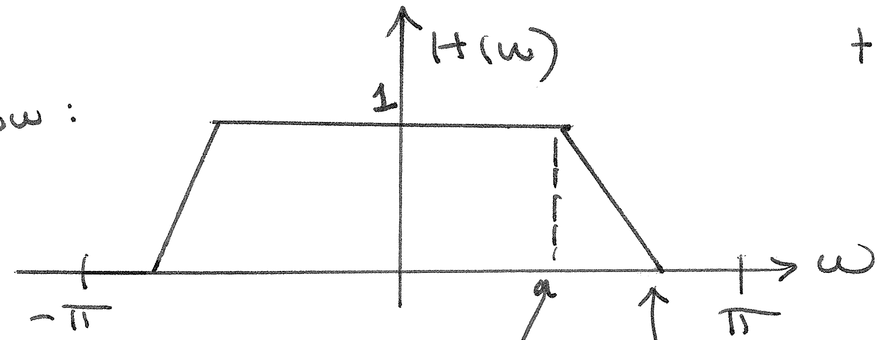
Problem 2 (c). You can continue your work for 2(c) here.

The A/D maps the passed frequencies to

$$(k\,2\pi)T_s = k\,2\pi\,\frac{1}{16} = k\,\frac{\pi}{8} \qquad k \in \{-8, \ldots, -1, 1, \ldots, 8\}$$

that is, $|k| \le 8$, $k \ne 0$

Now:



$$\frac{4\pi}{8} - \frac{\pi}{8} = \frac{3\pi}{8} \qquad \frac{4\pi}{8} + \frac{\pi}{8} = \frac{5\pi}{8}$$

The DT freqs that pass are: unity gain are:

$$k\,\frac{\pi}{8} \text{ for } |k| \le 3,\ k \ne 0 \qquad k \in \{-3, -2, -1, 1, 2, 3\}$$

and the frequency $4\frac{\pi}{8} = \frac{\pi}{2}$ gets a gain of $1/2$

as it passes thru the system $\Rightarrow k = 4$

Thus, final answer:

$$y[n] = \frac{1}{2}\,\frac{j(-1)^{-4}}{4\pi}\,e^{-j\frac{\pi}{2}n} + \sum_{\substack{k=-3 \\ k \ne 0}}^{3} \frac{j(-1)^k}{k\pi}\,e^{jk\frac{\pi}{8}n} + \frac{1}{2}\,\frac{j(-1)^4}{4\pi}\,e^{j\frac{\pi}{2}n}$$

Workout Problem 3

(a) Let $H_0(\omega)$ be the Discrete Time Fourier Transform (DTFT) of the impulse response $h_0[n]$ defined below.

$$h_0[n] = 2\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}\sin\left(\frac{\pi}{4}n\right) \tag{2}$$

Note that $h_0[n]$ is both real-valued and odd-symmetric as a function of time. Thus, $H_0(\omega)$ is purely imaginary-valued and odd-symmetric as a function of frequency. Plot the magnitude $H_0(\omega)$ over $-\pi < \omega < \pi$ in the space provided. The factor $j = \sqrt{-1}$ is included a scaling factor on the vertical axis.

(b) Determine and plot the DTFT $X(\omega)$ over $-\pi < \omega < \pi$ (in the space provided) of the signal $x[n]$ below:

$$x[n] = \frac{1}{2}\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} + \frac{1}{4}\left\{\frac{\sin\left(\frac{\pi}{2}(n-2)\right)}{\pi(n-2)} + \frac{\sin\left(\frac{\pi}{2}(n+2)\right)}{\pi(n+2)}\right\}$$

Hint: The maximum value of $X(\omega)$ is 1 and occurs at $\omega = 0$. Also, if it helps, two trigonometric identities: $\cos(\theta) = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$ and $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$.

(c) Determine and plot the Fourier Transform for the signal $y_0[n]$ defined below, where $\hat{x}_0[n] = x[n] * h_0[n]$ with $h_0[n]$ and $x[n]$ defined in parts (a) and (b), respectively. Plot $Y_0(\omega)$ over $-\pi < \omega < \pi$ in the space provided.

$$y_0[n] = x[n] + j\hat{x}_0[n] \quad \text{where:} \quad \hat{x}_0[n] = x[n] * h_0[n]$$

(d) Determine and plot the Fourier Transform for the signal $z_0[n]$ defined below where, as defined previously, $\hat{x}_0[n] = x[n] * h_0[n]$ with $h_0[n]$ and $x[n]$ defined in parts (a) and (b), respectively. Plot $Z_0(\omega)$ over $-\pi < \omega < \pi$ in the space provided.

$$z_0[n] = x[n]\cos\left(\frac{\pi}{2}n\right) - \hat{x}_0[n]\sin\left(\frac{\pi}{2}n\right)$$

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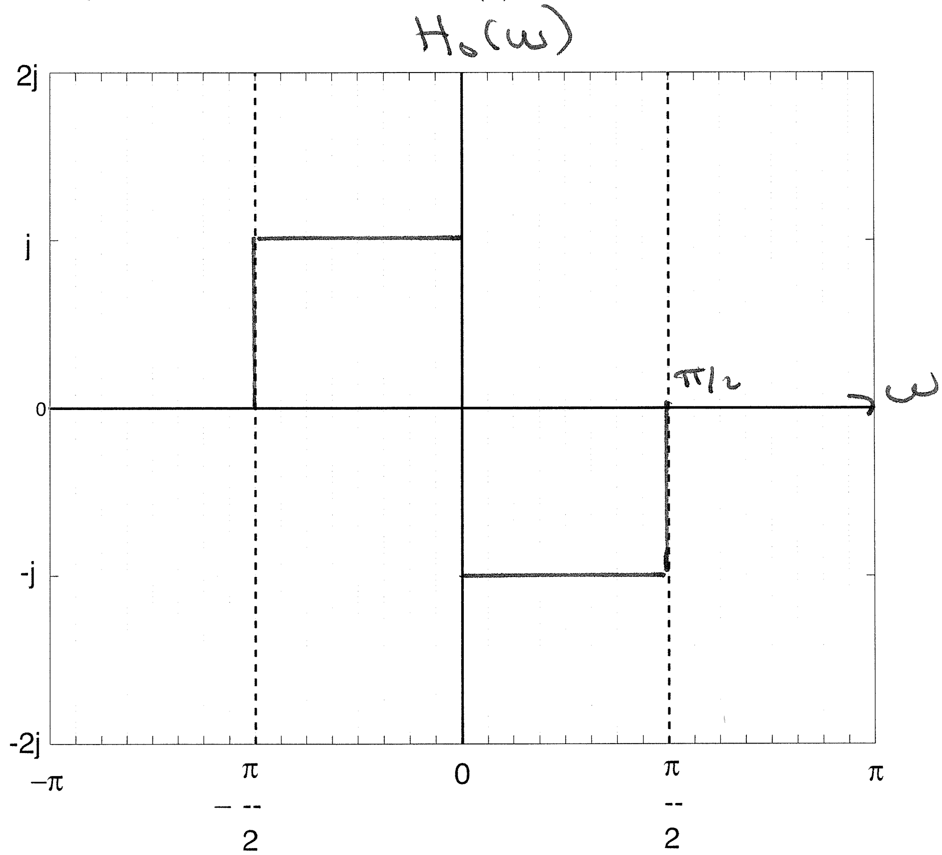
$$\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} \overset{DTFT}{\longleftrightarrow} \mathrm{rect}\left(\frac{\omega}{\pi/2}\right) \quad \text{over} \quad -\pi < \omega < \pi$$

$$2\left(\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}\right)\sin\left(\frac{\pi}{4}n\right) \overset{DTFT}{\longleftrightarrow} \frac{1}{j}\,\mathrm{rect}\left(\frac{\omega - \pi/4}{\pi/2}\right) - \frac{1}{j}\,\mathrm{rect}\left(\frac{\omega + \pi/4}{\pi/2}\right)$$

and $\frac{1}{j} = -j$

Plot your answer to Problem 3 (a) here. Show work above.

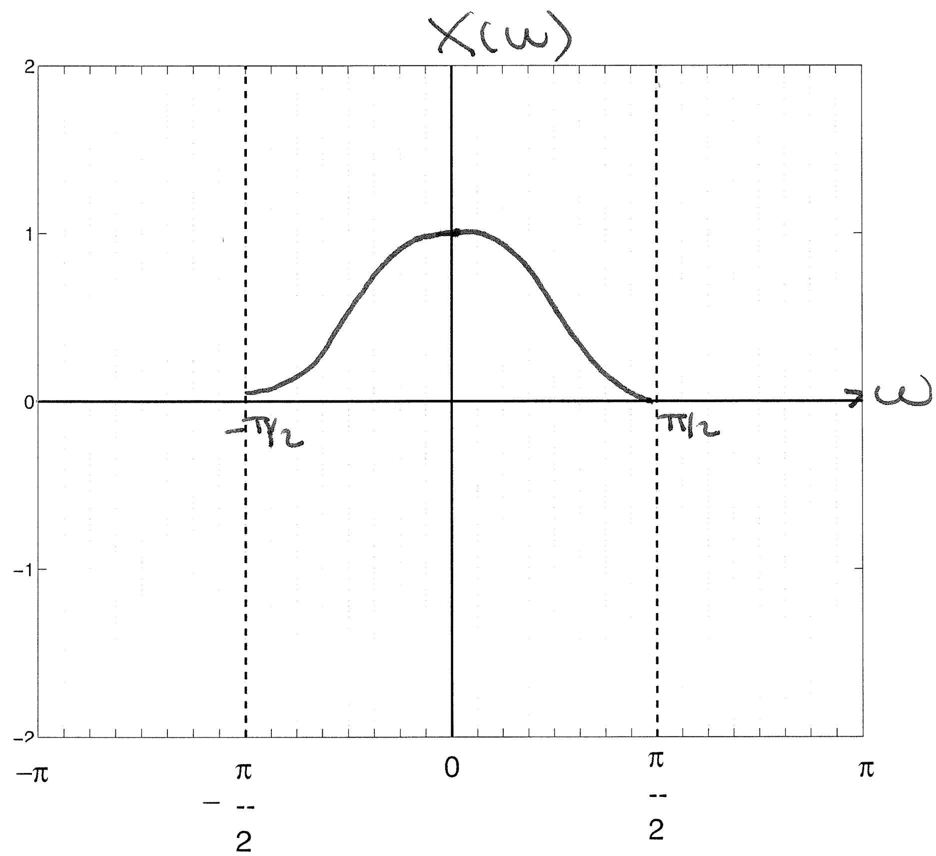


$$X(\omega) = \mathrm{rect}\left(\frac{\omega}{\pi}\right)\left\{\frac{1}{2} + \frac{1}{4}e^{-j2\omega} + \frac{1}{4}e^{j2\omega}\right\}$$

$$= \mathrm{rect}\left(\frac{\omega}{\pi}\right)\left\{\frac{1}{2} + \frac{1}{2}\cos(2\omega)\right\}$$

$$= \cos^2(\omega)\ \underbrace{\mathrm{rect}\left(\frac{\omega}{\pi}\right)}_{=1 \text{ for } |\omega| < \frac{\pi}{2}}$$

Plot your answer to Problem 3 (b) here. Show work above.

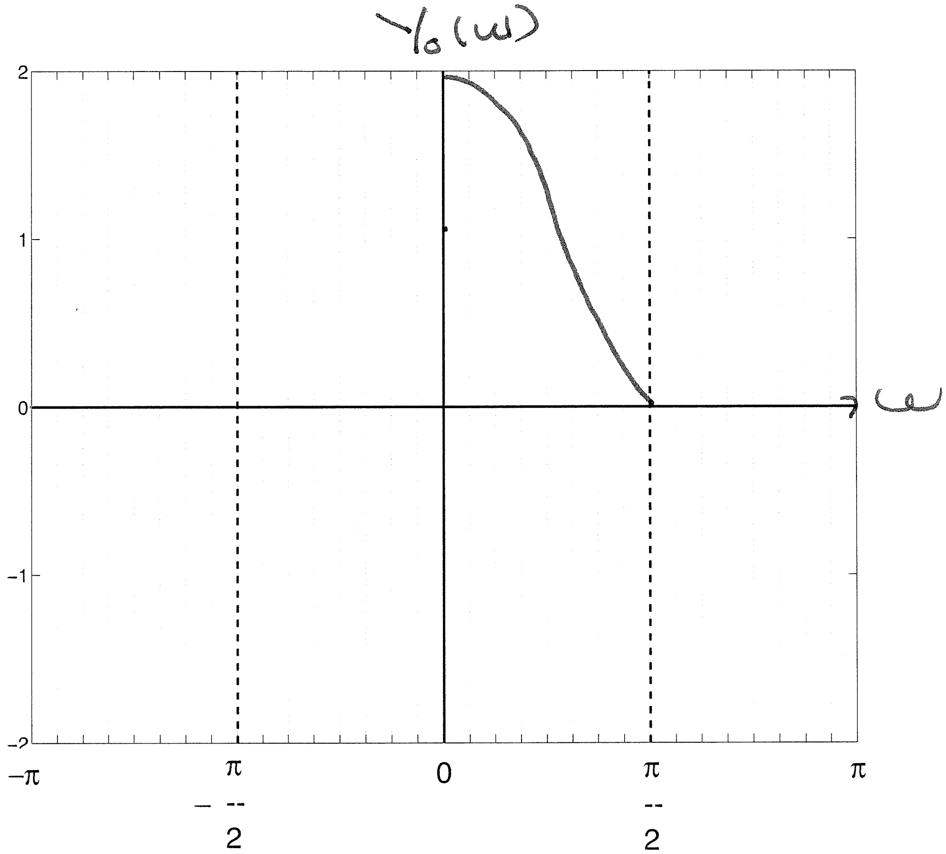


$$Y_0(\omega) = X(\omega) + j\,X(\omega)\,H_0(\omega)$$
$$= X(\omega)\{1 + j\,H_0(\omega)\}$$

only occupies $-\frac{\pi}{2} < \omega < \frac{\pi}{2}$

$$\begin{cases} 1 + j(j) = 0, & -\frac{\pi}{2} < \omega < 0 \\ 1 + j(-j) = 2, & 0 < \omega < \frac{\pi}{2} \end{cases}$$

Plot your answer to Problem 3 (c) here. Show work above.



$$z_0[n] = \mathrm{Re}\{y_0[n]\, e^{j\pi/2\, n}\}$$

$$z_0[n] = \mathrm{Re}\left\{ \left(x[n] + j\,\hat{x}_0[n]\right)\Big(\underbrace{e^{j\frac{\pi}{2}n}}_{\cos(\frac{\pi}{2}n)+j\sin(\frac{\pi}{2}n)}\Big)\right\}$$

and

$$y_0[n]\, e^{j\frac{\pi}{2}n} \overset{\mathcal{F}}{\longleftrightarrow} Y_0\left(\omega - \frac{\pi}{2}\right)$$

Define $\tilde{z}_0[n] = y_0[n]\, e^{j\frac{\pi}{2}n}$

$$\mathrm{Re}\{\tilde{z}_0[n]\} = \frac{1}{2}\tilde{z}_0[n] + \frac{1}{2}\tilde{z}_0^*[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{2}\tilde{Z}_0(\omega) + \frac{1}{2}\tilde{Z}_0^*(-\omega)$$

conjugate above doesn't factor into this problem since $X(\omega)$ is purely real-valued

Plot your answer to Problem 3 (d) here. Show work above.

$Z_0(\omega)$