Exam 3 Solution

(a) \( X_1(t) = T_s \cdot j \cdot 2\pi t \cdot \left\{ \frac{\sin(5t)}{\pi t} \right\}^2 \)

\[
= T_s \left\{ \frac{\sin(5t)}{\pi t} \right\} \cdot 2j \cdot \sin(5t)
\]

\[
= T_s \frac{\sin(5t)}{\pi t} e^{j \cdot 5t} - T_s \frac{\sin(5t)}{\pi t} e^{-j \cdot 5t}
\]

\[\omega_{\text{max}} = 10\]

\[
\omega_s = 40
\]

Since \( T_s = \frac{2\pi}{40} \)

\[40 > 2 \omega_{\text{max}} = 20\]

\( \Rightarrow \) no aliasing

Thus, DTFT \( X_d(\omega) = \frac{1}{T_s} \cdot X_1(F_s \omega) = \frac{1}{T_s} \cdot X_1(\frac{\omega}{T_s})\)

\[\omega_a = 10\] is mapped to \( \omega_d = 10 \cdot T_s = 10 \cdot \frac{2\pi}{40} = \frac{\pi}{2}\)

(b) \( T_s = \frac{2\pi}{15} \)

\[15 < 2(10) = 20 \Rightarrow \text{aliasing!}\]

The aliasing starts at \( (\omega_s - \omega_{\text{max}}) \cdot T_s \)

\[
= (15 - 10) \cdot \frac{2\pi}{15} = \frac{2\pi}{3}
\]

See next page for plots.
black = original Fourier Transform

red = replica centered at positive sampling rate = 15

blue = replica centered at negative sampling rate = -15

green = final answer = sum of all three from - half-sample-rate to + half sampling-rate demarcated by vertical lines

analog frequency (rads/sec)

digital frequency (rads/sample)
Part (c) \[ X_2(t) = -j T_s \left\{ \frac{\sin(10t)}{10t} \right\} \]

\[ \leftrightarrow \quad T_s \left( -j \omega \right) \text{rect}\left(\frac{\omega}{20}\right) \]

\[ = T_s \omega \text{rect}\left(\frac{\omega}{20}\right) = X_2(\omega) \]

\( \omega_{\text{max}} = 10 \quad \omega_s = 30 > 2(10) = 20 \Rightarrow \text{no aliasing} \)

\( \omega_{\text{max}} = 10 \) is mapped to \( 10 T_s = 10 \frac{2\pi}{30} = \frac{2\pi}{3} \)

(d) \( T_s = \frac{2\pi}{15} \quad \omega_s = 15 < 2(10) \Rightarrow \text{aliasing} \)

The aliasing starts at:

\[ (\omega_s - \omega_{\text{max}}) T_s = (15 - 10) \frac{2\pi}{15} = \frac{2\pi}{3} \]

See plots on next page
black = original Fourier Transform

red = replica centered at positive sampling rate = 15

blue = replica centered at negative sampling rate = -15

green = final answer = sum of all three from - half-sample-rate to + half sampling-rate demarcated by vertical lines

analog frequency (rads/sec)
digital frequency (rads/sample)
(e) \( X_{3a}(t) = \frac{j T_s}{\pi} \left\{ \frac{\sin(10(t-t_0))}{\pi(t-t_0)} - \frac{\sin(10(t+t_0))}{\pi(t+t_0)} \right\} \)

\( t_0 = \frac{T_s}{10} \)

\( X_{3a}(\omega) = T_s \text{rect} \left( \frac{\omega}{20} \right) \frac{1}{\pi} \left\{ e^{-j\frac{\omega T_s}{10}} + e^{j\frac{\omega T_s}{10}} \right\} \)

\( = T_s \sin \left( \frac{\omega \pi T_s}{10} \right) \text{rect} \left( \frac{\omega}{20} \right) \)

\( \omega_{\text{max}} = 10 \)

\( T_s = \frac{2\pi}{20} \quad \omega_s = 2\omega_{\text{max}} = 20 \quad \Rightarrow \text{Nyquist rate} \)

\( \omega_{\text{max}} = 0 \) mapped to \( 10 T_s = 10 \frac{2\pi}{20} = \pi \)

(f) \( T_s = \frac{2\pi}{15} \quad \omega_s = 15 \quad < 2 \omega_{\text{max}} = 20 \)

\( \Rightarrow \text{aliasing} \)

aliasing starts at \((\omega_s - \omega_{\text{max}}) T_s = \frac{7\pi}{15}\)

See plots on next page
black = original Fourier Transform

red = replica centered at positive sampling rate = 15

blue = replica centered at negative sampling rate = -15

green = final answer = sum of all three from - half-sample-rate to + half sampling-rate demarcated by vertical lines
\[ X_4(t) = T_s \frac{\pi}{2} \left\{ \frac{\sin(2t)}{\pi t} \frac{\sin(8t)}{\pi t} \right\} \]

\[ X_{d4}(\omega) \]

\[ \omega_{\text{max}} = 10 \]

\[ T_s = \frac{2\pi}{60} \]

\[ 60 \geq 2 \times (\omega_{\text{max}}) = 20 \implies \text{no aliasing} \]

\[ \omega_a = 5 \text{ is mapped to } 6 \quad \frac{2\pi}{60} = \frac{\pi}{5} \]

\[ \omega_a = 10 \text{ is mapped to } 10 \quad \frac{2\pi}{60} = \frac{\pi}{3} \]

\[ \frac{\omega}{T_s} \]

\[ \frac{\omega}{T_s} \]

\[ T_s = \frac{2\pi}{18} \]

\[ 18 < 20 \implies \text{aliasing starts at} \]

\[ (\omega_s - \omega_{\text{max}})T_s = (18 - 10) \frac{2\pi}{18} = \frac{8\pi}{9} \]

See plots on next page

\[ T_s = \frac{2\pi}{16} \]

\[ 16 < 20 \implies \text{aliasing starts at} \]

\[ (\omega_s - \omega_{\text{max}})T_s = (16 - 10) \frac{2\pi}{16} = \frac{3\pi}{4} \]

See plots on next page after next page
black = original Fourier Transform

red = replica centered at positive sampling rate = 18

blue = replica centered at negative sampling rate = -18

green = final answer = sum of all three from - half-sample-rate to + half sampling-rate demarcated by vertical lines
black = original Fourier Transform

green = final answer = sum of all three from - half-sample-rate to + half sampling-rate demarcated by vertical lines
(j) \( T_5 = \frac{2\pi}{48} \)

Six analog frequencies are mapped to:

1. \( \frac{\pi}{8} = 0.3927 \)
2. \( \frac{3\pi}{8} = 0.9812 \)
3. \( \frac{5\pi}{8} = 1.9625 \)
4. \( \frac{7\pi}{8} = 2.9438 \)
5. \( \frac{9\pi}{8} = 3.9251 \)
6. \( \frac{11\pi}{8} = 4.9064 \)

All less than \( \pi \), so no aliasing.
\[ T_5 = \frac{2\pi}{2} \quad \omega_5 = 2\pi \]

\[ \frac{2\pi}{2} = \frac{\pi}{2} \]

\[ \frac{2\pi}{2} = \pi \]

\[ \frac{2\pi}{2} = \frac{3\pi}{4} \quad \Rightarrow \quad \frac{3\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4} \]

\[ \cos \left( \frac{-3\pi}{4} \right) = \cos \left( \frac{3\pi}{4} \right) \]

\[ \frac{2\pi}{2} = \frac{3\pi}{4} \quad \Rightarrow \quad \frac{3\pi}{2} - \frac{4\pi}{2} = -\frac{\pi}{2} \]

\[ \cos \left( \frac{-\pi}{4} \right) = \cos \left( \frac{\pi}{4} \right) \]

\[ \frac{2\pi}{2} = \frac{7\pi}{4} \quad \Rightarrow \quad \frac{7\pi}{4} - \frac{8\pi}{4} = -\frac{\pi}{4} \]

\[ \cos \left( \frac{-\pi}{4} \right) = \cos \left( \frac{\pi}{4} \right) \]

24 \cdot \frac{2\pi}{2} = 2\pi \Rightarrow \text{aliased to zero}