

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(\omega)$
		$y(t)$	$Y(\omega)$
4.3.0	Duality	$X(t)$	$2\pi x(-\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(\omega) + bY(\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
4.3.3	Conjugation	$x^*(t)$	$X^*(-\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(\omega)Y(\omega)$
4.5	Multiplication	$x(t)y(t) \xleftrightarrow{F} \frac{1}{2\pi} X(\omega) * Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\theta)Y(\omega - \theta) d\theta$	
4.3.4	Differentiation in Time	$\frac{d}{dt} x(t)$	$j\omega X(\omega)$
4.3.4	Integration	$\int_{-\infty}^t x(t) dt$	$\frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j \frac{d}{d\omega} X(\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\left\{ \begin{array}{l} X(\omega) = X^*(-\omega) \\ \text{Re}\{X(\omega)\} = \text{Re}\{X(-\omega)\} \\ \text{Im}\{X(\omega)\} = -\text{Im}\{X(-\omega)\} \\ X(\omega) = X(-\omega) \\ \angle X(\omega) = -\angle X(-\omega) \end{array} \right.$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}\{x(t)\}$ [x(t) real] $x_o(t) = \mathcal{O}\{x(t)\}$ [x(t) real]	$\text{Re}\{X(\omega)\}$ $j\text{Im}\{X(\omega)\}$
Initial Value Theorems:		$x(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) d\omega$	
		$X(0) = \int_{-\infty}^{+\infty} x(t) dt$	
4.3.7	Parseval's Relation for Aperiodic Signals		$\int_{-\infty}^{+\infty} x(t) ^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) ^2 d\omega$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi \delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0, \text{ otherwise}$
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0, \text{ otherwise}$
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0, \text{ otherwise}$
$x(t) = 1$	$2\pi \delta(\omega)$	$a_0 = 1, a_k = 0, k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave		
$x(t) = \begin{cases} 1, & t < T_1 \\ 0, & T_1 < t \leq \frac{T}{2} \end{cases}$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
and $x(t + T) = x(t)$		
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T} \text{ for all } k$
$x(t) = \begin{cases} 1, & t < T_1 \\ 0, & t > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	— $\text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{F} \frac{\sin(\pi \frac{\omega}{2})}{\frac{\omega}{2}}$
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \omega < W \\ 0, & \omega > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi \delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at} u(t), \text{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at} u(t), \text{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t), \text{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Property	Aperiodic Signal	Fourier Transform
	$x[n]$	$X(\omega)$
	$y[n]$	$Y(\omega)$
Linearity	$ax[n] + by[n]$	$aX(\omega) + bY(\omega)$
Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0} X(\omega)$
Frequency Shifting	$e^{j\omega_0 n} x[n]$	$X(\omega - \omega_0)$
Conjugation	$x^*[n]$	$X^*(-\omega)$
Time Reversal	$x[-n]$	$X(-\omega)$
Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(\frac{\omega}{k})$
Convolution	$x[n] * y[n]$	$X(\omega)Y(\omega)$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi} \int_{2\pi} X(\theta)Y(\omega - \theta) d\theta$
Differencing in Time	$x[n] - x[n - 1]$	$(1 - e^{-j\omega})X(\omega)$
Accumulation	$\sum_{k=-\infty}^n x[k]$	$\frac{1}{1 - e^{-j\omega}} X(\omega)$ $+ \pi X(0) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
Differentiation in Frequency	$nx[n]$	$j \frac{dX(\omega)}{d\omega}$
Conjugate Symmetry for	$x[n]$ real	$\begin{cases} X(\omega) = X^*(-\omega) \\ \text{Re}\{X(\omega)\} = \text{Re}\{X(-\omega)\} \\ \text{Im}\{X(\omega)\} = -\text{Im}\{X(-\omega)\} \end{cases}$

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

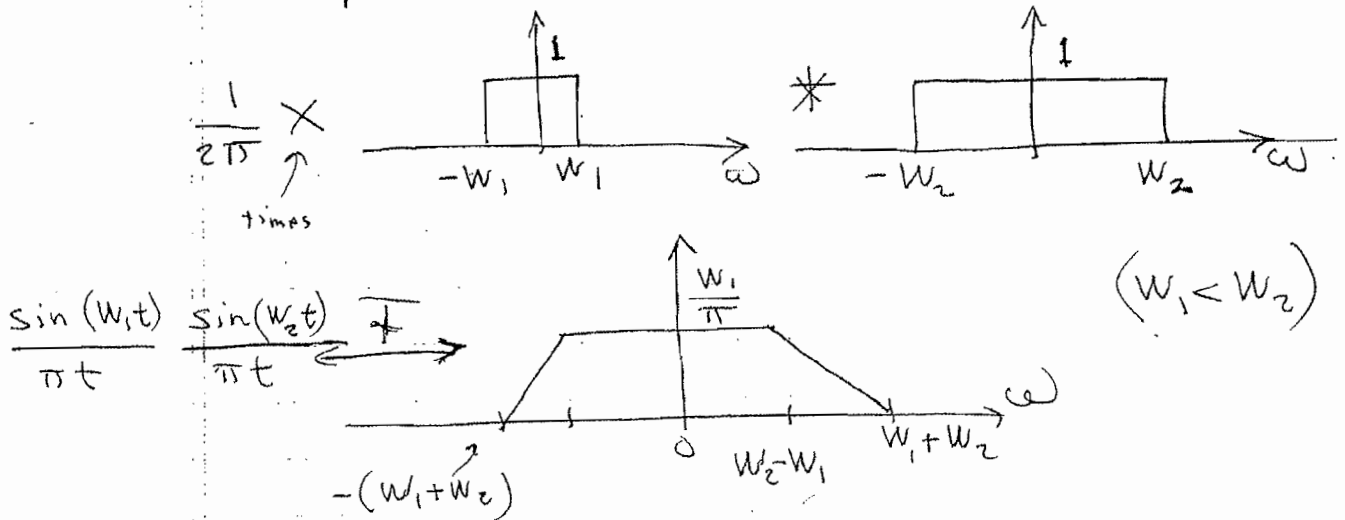
Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=(N)} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational \Rightarrow The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational \Rightarrow The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \dots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational \Rightarrow The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & n \leq N_1 \\ 0, & N_1 < n \leq N/2 \end{cases}$ and $x[n + N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, k \neq 0, \pm N, \pm 2N, \dots$ $a_k = \frac{2N_1 + 1}{N}, k = 0, \pm N, \pm 2N, \dots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], a < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & n \leq N_1 \\ 0, & n > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \text{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \leq \omega \leq W \\ 0, & W < \omega \leq \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n + 1)a^n u[n], a < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n + r - 1)!}{n!(r - 1)!} a^n u[n], a < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

Some Fourier Transform Results involving sinc functions:

- Two sinc functions multiplied in time:

$$\frac{\sin(\omega_1 t)}{\pi t} \frac{\sin(\omega_2 t)}{\pi t} \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} \mathcal{F} \left\{ \frac{\sin(\omega_1 t)}{\pi t} \right\} * \mathcal{F} \left\{ \frac{\sin(\omega_2 t)}{\pi t} \right\}$$

multiplication in time \rightarrow convolution in frequency



- Two sinc functions convolved in time:

$$\frac{\sin(\omega_1 t)}{\pi t} * \frac{\sin(\omega_2 t)}{\pi t} \xleftrightarrow{\mathcal{F}} \mathcal{F} \left\{ \frac{\sin(\omega_1 t)}{\pi t} \right\} \times \mathcal{F} \left\{ \frac{\sin(\omega_2 t)}{\pi t} \right\}$$

$\omega_1 < \omega_2$

convolution in time \rightarrow multiplication in frequency

Diagram illustrating the multiplication of two rectangular pulses in the frequency domain. The resulting spectrum is a narrower rectangular pulse with a height of 1, centered at 0. The x-axis is labeled ω and has points $-\omega_1$, ω_1 marked. A note "times" is present.

THUS:
$$\frac{\sin(\omega_1 t)}{\pi t} * \frac{\sin(\omega_2 t)}{\pi t} = \frac{\sin(\omega_1 t)}{\pi t}$$

$\omega_1 < \omega_2$