SOLUTION

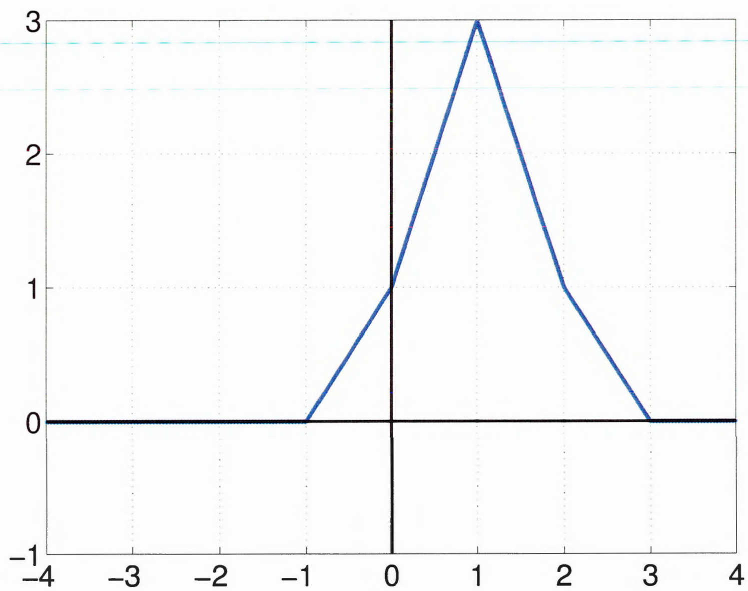
EE301 Signals and Systems **Spring 2019**
Exam 2 **Thursday, 12:00-1:25 pm, Mar. 28, 2019**

Cover Sheet

Test Duration: 75 minutes.
Coverage: Chapters 3-4, Hmwks 6-7
Open Book but Closed Notes.
One 8.5 in. x 11 in. crib sheet (both sides)
Calculators NOT allowed.
All work should be done on the sheets provided.
You can NOT do work on the back of a page unless permission is granted.
No work on the back of a page will be graded unless permission is granted.
You must show all work for each problem to receive full credit.

Additional Directions

There are four problems, each with multiple parts.
Do each part in order and clearly mark your answer for each part.



Problem 1. Consider the time-domain signal $x(t)$ plotted above. Using the properties of the Fourier Transform, you are able to compute all of the quantities required in each of the parts below without having to determine the corresponding Fourier Transform, $X(\omega)$. Indicate which properties you are using and clearly label your final answer for each part.

(a) Find $X(0)$. (Find the value of the Fourier Transform at $\omega = 0$.)

(b) Evaluate $\int_{-\infty}^{\infty} X(\omega)\,d\omega$.

(c) Evaluate $\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$.

(d) Plot the phase of the Fourier Transform, $\angle X(\omega)$. That is, $X(\omega) = |X(\omega)|e^{\angle X(\omega)}$; plot $\angle X(\omega)$

(e) Find the value below, where $H(\omega) = 2\dfrac{\sin(\omega)}{\omega}$

$$\int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j2\omega}\,d\omega = ??$$

(a) $X(0) = \int_{-\infty}^{\infty} x(t)\,dt = 2\int_{0}^{2} x(t+1)\,dt$

$= 2\left\{(1)(1)+\frac{1}{2}(1)2+\frac{1}{2}(1)(1)\right\}$

$= 2\left\{1+1+\frac{1}{2}\right\} = 4+1 = 5$

Additional Space for Problem 1 answer and work.

(b) $\displaystyle\int_{-\infty}^{\infty} X(\omega)\,d\omega = 2\pi\, x(t)\Big|_{t=0} = 2\pi(1) = 2\pi$

(c) $\displaystyle\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega = 2\pi \int_{-\infty}^{\infty} x^2(t)\,dt$

$\displaystyle = 2\pi\,(2)\int_0^2 x^2(t+1)\,dt$

$\displaystyle = 4\pi\left\{\int_0^1 (-2t+3)^2\,dt + \int_1^2 (-1)^2(t-2)^2\,dt\right\}$

$\displaystyle = 4\pi\left\{\frac{1}{-2}\,\frac{(-2t+3)^3}{3}\Big]_0^1 + \frac{(t-2)^3}{3}\Big]_1^2\right\}$

$\displaystyle = 4\pi\left\{-\frac{1}{2}\left(\frac{1}{3}-\frac{27}{3}\right) + \frac{1}{3}\right\}$

$\displaystyle = 4\pi\left\{\frac{13}{3}+\frac{1}{3}\right\} = 4\pi\,\frac{14}{3}$

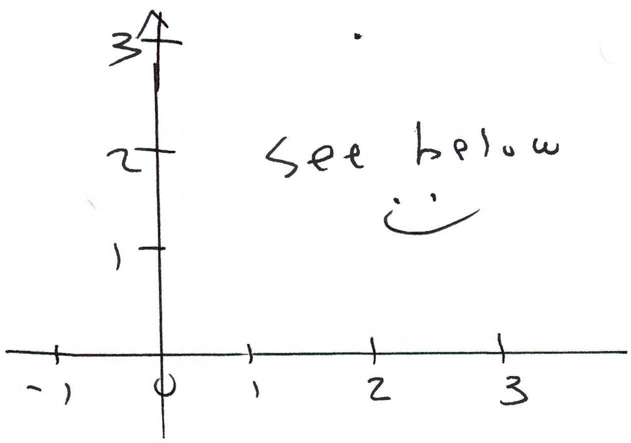
(d) $\angle X(\omega) = -\omega$ line with slope $= -1$ passes thru origin

Additional Space for Problem 1 answer and work.

(e) $\frac{1}{2\pi}\int X(\omega)H(\omega)e^{j\omega 2}\,d\omega = y(t)\Big|_{t=2}$

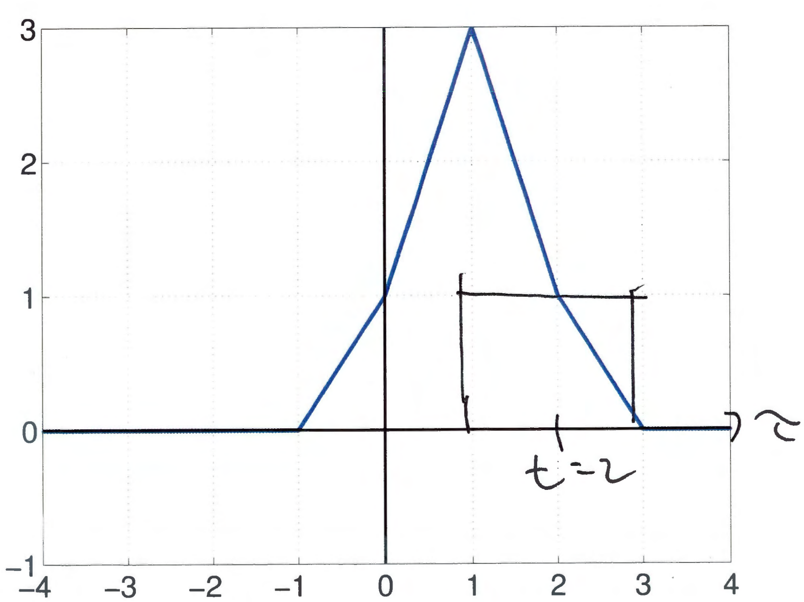
where: $y(t) = x(t) * h(t)$

$= x(t) * \text{rect}\left(\frac{t}{2}\right)$



area under product computed in part (a) as $\frac{5}{2}$

final answer: $2\pi \frac{5}{2} = 5\pi$



Problem 2 You are given the Fourier Transform pair below, which was derived in homework. Show all work and clearly mark your answers for parts (a), (b), and (c) in the space provided.

$$x(t) = \{1 + \cos(\pi t)\}\,\mathrm{rect}\left(\frac{t}{2}\right) \longleftrightarrow X(\omega) = 2\frac{\sin(\omega)}{\omega}\frac{\pi^2}{\pi^2 - \omega^2}$$

(a) Determine the Fourier Transform, $X_a(\omega)$, of the derivative of $x(t)$ using the derivative property.

$$x_a(t) = \frac{d}{dt}x(t) = -\pi\,\sin(\pi t)\,\mathrm{rect}\left(\frac{t}{2}\right)$$

(b) Determine the Fourier Transform, $X_b(\omega)$, of the signal below (same as $x_a(t)$) but using the Fourier Transform of a rectangular pulse, linearity, and (one of) the handwritten frequency shift properties at the bottom of the Table of Fourier Transform properties attached.

$$x_b(t) = -\pi\,\mathrm{rect}\left(\frac{t}{2}\right)\sin(\pi t)$$

(c) Show that the two answers are the same: $X_a(\omega) = X_b(\omega)$

(a) $X_a(\omega) = j\omega X(\omega) = 2j\sin(\omega)\dfrac{\pi^2}{\pi^2 - \omega^2}$

(b) $X_b(\omega) = -\pi\dfrac{1}{2j}\dfrac{\sin(\omega - \pi)}{(\omega - \pi)/2} + \pi\dfrac{1}{2j}\dfrac{\sin(\omega + \pi)}{(\omega + \pi)/2}$

$= -\pi j\sin(\omega)\left\{\dfrac{1}{\omega - \pi} - \dfrac{1}{\omega + \pi}\right\}$

$= -\pi j\sin(\omega)\left\{\dfrac{\omega + \pi - (\omega - \pi)}{\omega^2 - \pi^2}\right\}$

$= 2j\sin(\omega)\dfrac{\pi^2}{\pi^2 - \omega^2}$ $\Big\}$ same

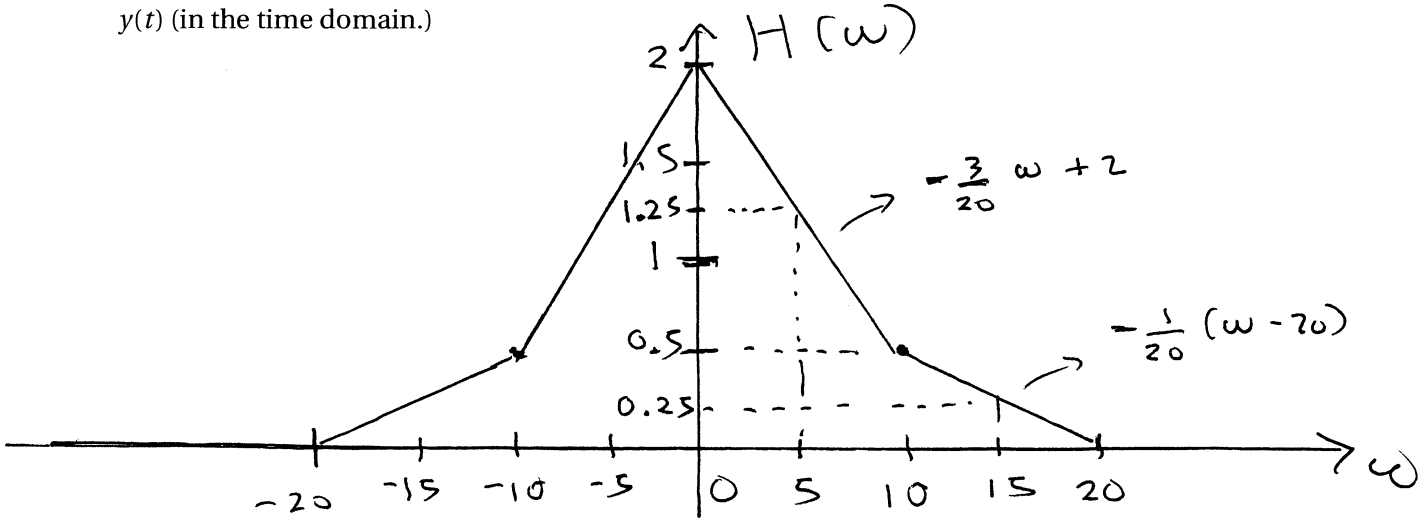
Problem 3. The signal

$$x(t) = 2 + 3\cos(5t) + 4\cos(10t) + 5\cos(15t) + 6\cos(20t + \pi/3) + 7\cos(25t - \pi/6) + 8\cos(30t - \pi/7)$$

is input to an LTI system with impulse response

$$h(t) = \frac{\pi}{10}\left\{\frac{\sin(10t)}{\pi t}\right\}^2 + \frac{\pi}{5}\left\{\frac{\sin(5t)}{\pi t}\right\}^2$$

Determine the output $y(t) = x(t) * h(t)$. Show all work and clearly indicate your final answer for $y(t)$ (in the time domain.)



$y(t) = (2)(2)$

$+ (1.25)\, 3\cos(5t)$

$+ (0.5)\, 4\cos(10t)$

$+ (0.25)\, 5\cos(15t)$

Problem 4. Determine the Fourier Transform of the full-wave rectified sinewave below.

$$y(t) = \left|\cos\left(\frac{\pi}{2}t\right)\right|$$

HINT: Express $y(t)$ as $y(t) = x(t)\cos\left(\frac{\pi}{2}t\right)$ where $x(t)$ is a train of rectangular pulses that is periodic with period $T = 4$ (s) and has a value $x(t) = 1$ for $-1 < t < +1$ and a value $x(t) = -1$ for $1 < t < 3$, such that the negative half-cycles of the sinewave are flipped in sign.

Show all work and which properties and pairs you use. You do NOT need to plot your final answer; just provide the expression for the Fourier Transform $Y(\omega)$.

$$y(t) = x(t)\cos\left(\frac{\pi}{2}t\right) \quad \Big\}\ \cos\left(2\pi\frac{1}{4}t\right) \Big\}\ \text{period} = 4$$

$$\xleftrightarrow{\mathcal{F}} Y(\omega) = \frac{1}{2}X\left(\omega - \frac{\pi}{2}\right) + \frac{1}{2}X\left(\omega + \frac{\pi}{2}\right)$$

$$x(t) = \sum_{k=-\infty}^{\infty} z(t - k4) = \sum_{k=-\infty}^{\infty} \tilde{a}_k e^{jk\frac{2\pi}{4}t}$$

where: $z(t) = \text{rect}\left(\frac{t}{2}\right) - \text{rect}\left(\frac{t-2}{2}\right)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{4}t} - \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{4}(t-2)}$$

$$= \sum_{k=-\infty}^{\infty} \{a_k - (-1)^k a_k\} e^{jk\frac{2\pi}{4}t}$$

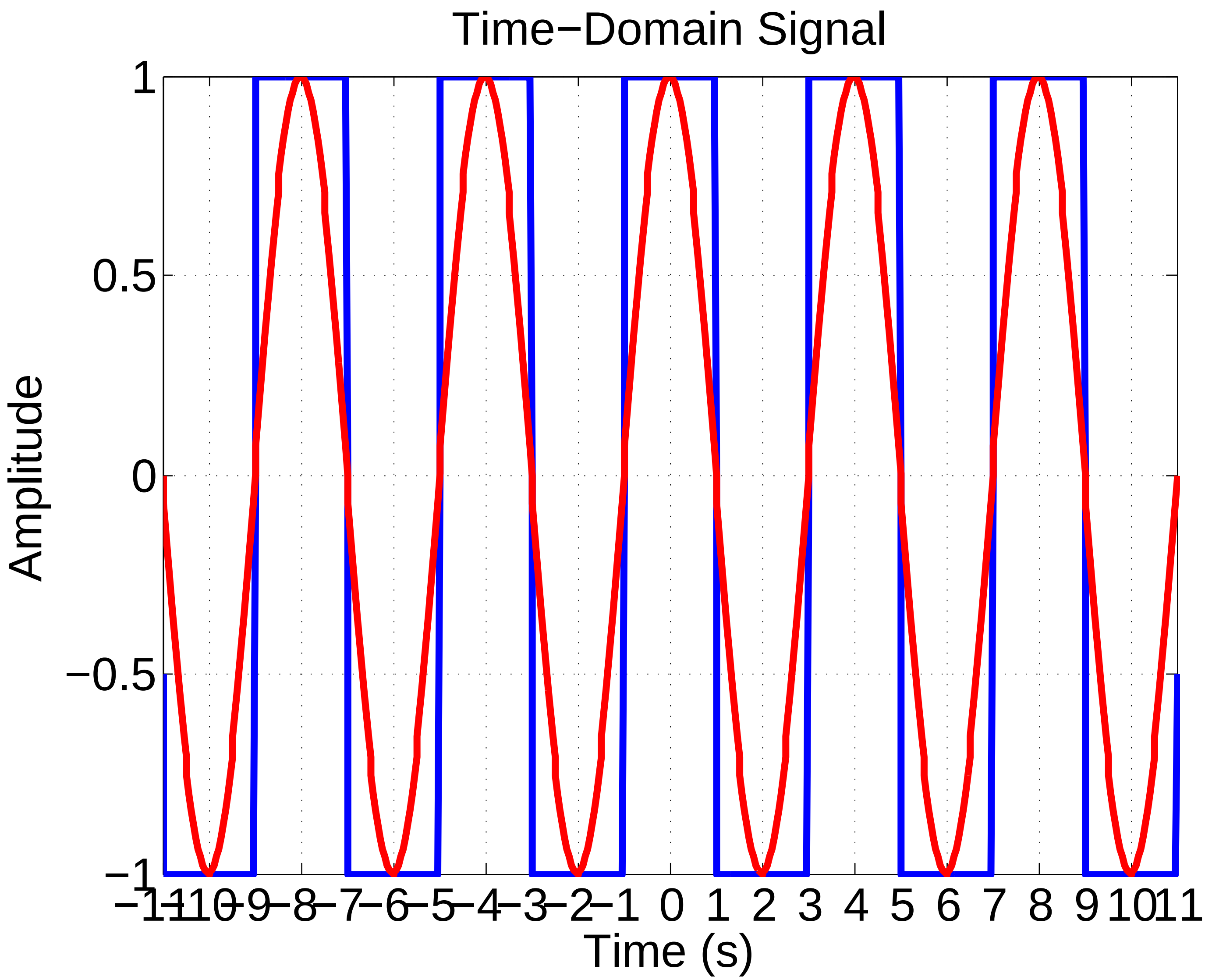
$$= \sum_{k=-\infty}^{\infty} a_k \underbrace{\left(1 - (-1)^k\right)}_{\substack{=0 \quad k \text{ even} \\ =2 \quad k \text{ odd}}} e^{jk\frac{2\pi}{4}t}$$

$$= \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} 2\,\frac{\sin\left(k\frac{\pi}{2}\right)}{k\pi} e^{jk\frac{2\pi}{4}t}$$

$$\xleftrightarrow{\mathcal{F}} \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} 4\pi\,\frac{\sin\left(k\frac{\pi}{2}\right)}{k\pi}\,\delta\left(\omega - k\frac{\pi}{2}\right)$$

$$a_k = \frac{\sin\left(k\pi\frac{2}{4}\right)}{k\pi} = \frac{\sin\left(k\frac{\pi}{2}\right)}{k\pi} = \frac{+1}{k\pi} \text{ or } \frac{-1}{k\pi} \text{ for } k \text{ odd}, \quad = 0 \text{ for } k \text{ even}$$

$$Y(\omega) = 2\sum_{k \text{ odd}} \frac{\sin\left(k\frac{\pi}{2}\right)}{k}\left\{\delta\left(\omega - (k+1)\frac{\pi}{2}\right) + \delta\left(\omega - (k-1)\frac{\pi}{2}\right)\right\}$$


Time−Domain Signal
Amplitude
Time (s)
1
0.5
0
−0.5
−1
−11 −10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9 10 11