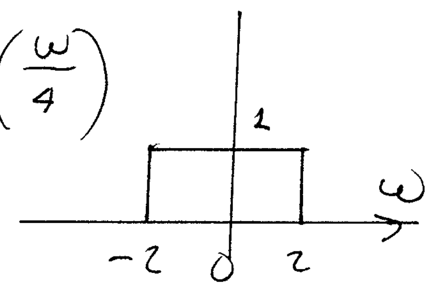
Exam 2 SP'08 ECE 301 SOLUTIONS ①

<u>Sol'n to Prob. 1</u> $x(t) = \pi t \left(\frac{\sin(2t)}{\pi t}\right)^2$

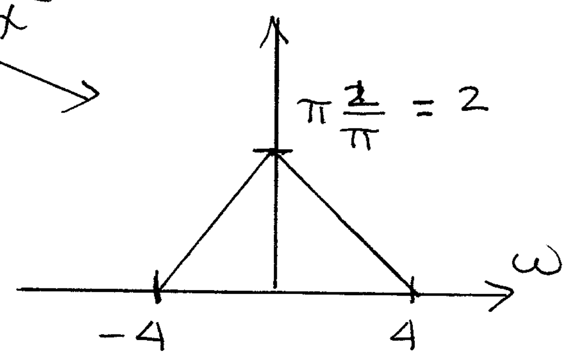
(a) At least two ways to solve this problem.

<u>Approach 1</u>:

$$\frac{\sin(2t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}\left(\frac{\omega}{4}\right)$$



$$\underbrace{\pi\left(\frac{\sin(2t)}{\pi t}\right)\left(\frac{\sin(2t)}{\pi t}\right)}_{W_1 = W_2 = 2} \overset{\mathcal{F}}{\longleftrightarrow}$$

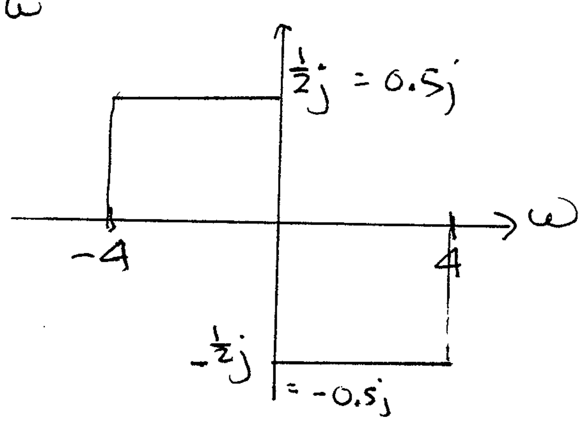


Now,

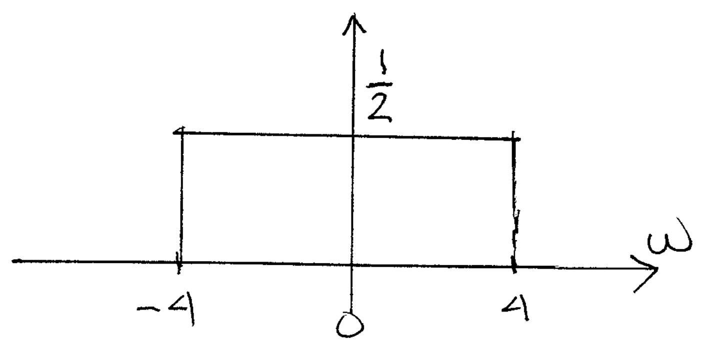
$$t\, x(t) \overset{\mathcal{F}}{\longleftrightarrow} j \frac{d}{d\omega} X(\omega)$$

THUS:

$$\pi t \left(\frac{\sin(2t)}{\pi t}\right)^2 \overset{\mathcal{F}}{\longleftrightarrow}$$



THUS: $|X(\omega)|$



Prob. 1 Sol'n (cont.)

Approach 2:

$$\pi t \left(\frac{\sin(2t)}{\pi t}\right)^2 = \left(\frac{\sin(2t)}{\pi t}\right) \sin(2t)$$

and use modulation property

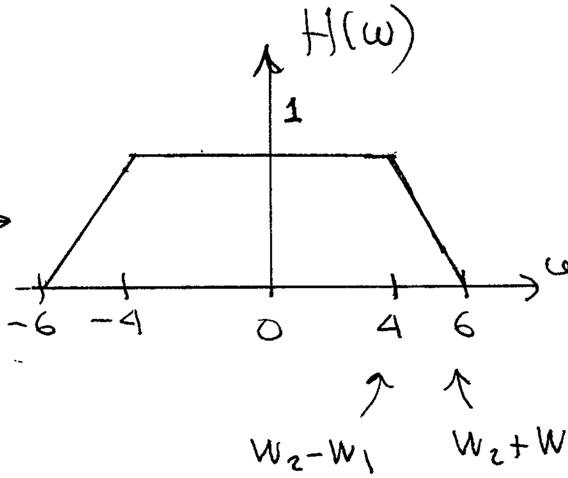
=> obtain same result

(b) $$\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}(4)\left(\frac{1}{2}\right)^2 = \frac{1}{2\pi}$$

Prob. 2

$$h(t) = \pi \frac{\sin(t)}{\pi t}\,\frac{\sin(5t)}{\pi t} \xleftrightarrow{\mathcal{F}}$$



$W_1 = 1 \qquad W_2 = 5$

(a)

$$x_1(t) = \cos(6t) \rightarrow \boxed{H(\omega)} \rightarrow y_1(t) = |H(6)| \cdot \cos(6t + \angle H(6))$$

$$= 0$$

(b) $k=0$ term is zero for all t

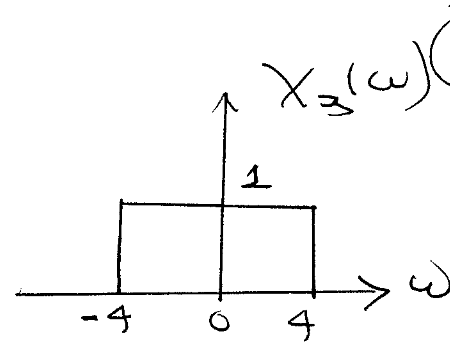
$$x_2(t) = \left(\frac{1}{2}\right)\sin(3t) + \left(\frac{1}{4}\right)\sin(6t) + \left(\frac{1}{8}\right)\sin(9t) + \ldots$$

only $\omega = 3$ makes it thru filter

$$y_2(t) = \frac{1}{2}\sin(3t)$$

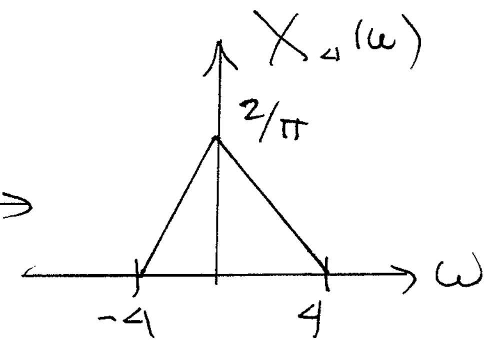
Prob. 2 Sol'n (cont.)

(c) $x_3(t) = \dfrac{\sin(4t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow}$



THUS: $Y_3(\omega) = X_3(\omega)\, H(\omega) = X_3(\omega)$

$$y_3(t) = \frac{\sin(4t)}{\pi t}$$



(d) $x_4(t) = \left(\dfrac{\sin(2t)}{\pi t}\right)^2 \overset{\mathcal{F}}{\longleftrightarrow}$

$$Y_4(\omega) = X_4(\omega)\, H(\omega) = X_4(\omega)$$

$$y_4(t) = \left(\frac{\sin(2t)}{\pi t}\right)^2$$

Prob. 3 Sol'n.

$$x(t) = \frac{2(2)}{2^2 + t^2} \Rightarrow a = 2$$

Text example:

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{a^2 + \omega^2}$$

Duality:

$$\frac{2a}{a^2 + \omega^2} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, e^{-a|-\omega|} = 2\pi e^{-a|\omega|}$$

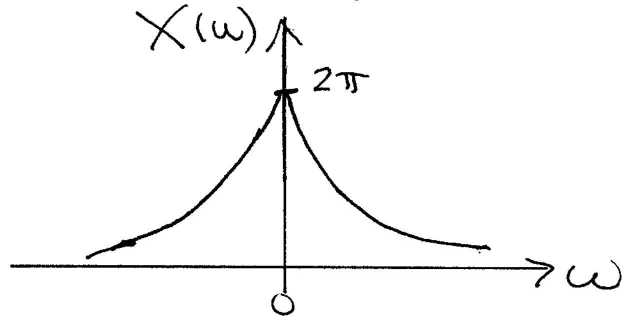
(a) THUS: $\dfrac{4}{4 + t^2} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi e^{-2|\omega|}$

(b) $x(t)$ real-valued and symmetric $\Rightarrow$

$X(\omega)$ real-valued and symmetric

Prob. 3 Sol'n (cont.)

(c)



(d) $E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

$= \frac{2}{2\pi}\int_0^{\infty} |X(\omega)|^2 d\omega = \frac{1}{\pi}\int_0^{\infty} \left(e^{-2\omega}\right)^2_{\wedge\,(2\pi)^2} d\omega$

$= 4\pi\left(-\frac{1}{4} e^{-4\omega}\Big|_0^{\infty}\right)$

$= 4\pi\left(-\frac{1}{4}\right)\left(e^{-\infty} - 1\right) = \pi$

$E_x = \pi$

(e) $y(t) = x(t)\sqrt{2}\cos(10t)$

$Y(\omega) = \frac{\sqrt{2}}{2} X(\omega - 10) + \frac{\sqrt{2}}{2} X(\omega + 10)$

$= \frac{1}{\sqrt{2}}\, 2\pi e^{-2|\omega-10|} + \frac{1}{\sqrt{2}}\, 2\pi\, e^{-2|\omega+10|}$

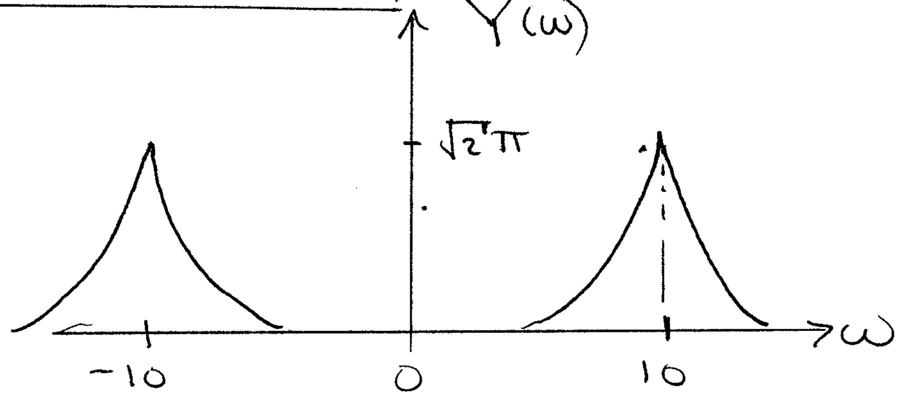
$= \sqrt{2}\pi\, e^{-2|\omega-10|} + \sqrt{2}\pi\, e^{-2|\omega+10|}$

(f) Since $\cos(10t)$ is also even $\Rightarrow$

$y(t)$ is real and even $\Rightarrow Y(\omega)$ is real and even

Prob. 3 Sol'n (cont.)

(g)



assume both terms are essentiall 0 at $\omega = 0$ for (h)

(h)

$$E_y = \int_{-\infty}^{\infty} y^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega$$

$$= \frac{2}{2\pi}\int_{0}^{\infty} |Y(\omega)|^2 d\omega$$

Now, since the max height of $Y(\omega)$ is $\frac{1}{\sqrt{2}}$ times the max height of $X(\omega) \Rightarrow$ max height of $|Y(\omega)|^2$ is $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$ times max height of $|X(\omega)|^2$

THUS: Area under $|Y(\omega)|^2$

$$= 2 \times \frac{1}{2} \times \text{area under } |X(\omega)|^2$$

$E_y = E_x = \pi \Rightarrow$ same energy!