ECE301 Signals and Systems
Spring 2005
Exam I
Solutions

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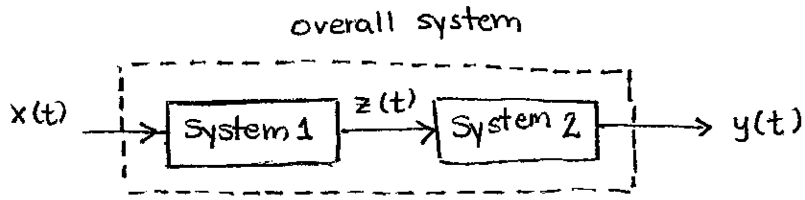
1.



System 1 : $z(t) = \int_{t-2}^{t} x(\tau)\, d\tau$

System 2 : $y(t) = \int_{t-4}^{t} z(\tau)\, d\tau$

overall system

$$y(t) = \int_{t-4}^{t} z(\tau)\, d\tau = \int_{t-4}^{t} \int_{\tau-2}^{\tau} x(\lambda)\, d\lambda\, d\tau$$

(a) Linear?

$$x_1(t) \rightarrow y_1(t) = \int_{t-4}^{t} \int_{\tau-2}^{\tau} x_1(\lambda)\, d\lambda\, d\tau$$

$$x_2(t) \rightarrow y_2(t) = \int_{t-4}^{t} \int_{\tau-2}^{\tau} x_2(\lambda)\, d\lambda\, d\tau$$

$$x_3(t) = a x_1(t) + b x_2(t) \longrightarrow y_3(t) = \int_{t-4}^{t} \int_{\tau-2}^{\tau} x_3(\lambda)\, d\lambda\, d\tau$$

$$= \int_{t-4}^{t} \int_{\tau-2}^{\tau} [a x_1(\lambda) + b x_2(\lambda)]\, d\lambda\, d\tau$$

$$= a \int_{t-4}^{t} \int_{\tau-2}^{\tau} x_1(\lambda)\, d\lambda\, d\tau + b \int_{t-4}^{t} \int_{\tau-2}^{\tau} x_2(\lambda)\, d\lambda\, d\tau$$

$$= a\, y_1(t) + b\, y_2(t)$$

$\therefore$ The overall system is linear.

(b) Time-Invariant?

$$x_1(t) \longrightarrow \boxed{S} \longrightarrow y_1(t) = \int_{t-4}^{t} \int_{\tau-2}^{\tau} x_1(\lambda)\, d\lambda\, d\tau$$

$$x_2(t) = x_1(t-t_0) \longrightarrow \boxed{S} \longrightarrow y_2(t) = \int_{t-4}^{t} \int_{\tau-2}^{\tau} x_2(\lambda)\, d\lambda\, d\tau$$

$$= \int_{t-4}^{t} \int_{\tau-2}^{\tau} x_1(\lambda - t_0)\, d\lambda\, d\tau$$

$$= \int_{\tau-2}^{\tau} \int_{t-4}^{t} x_1(\lambda - t_0)\, d\tau\, d\lambda$$

Let $\lambda - t_0 = \alpha$

$$= \int_{\tau-2}^{\tau} \int_{t-4-t_0}^{t-t_0} x_1(\alpha)\, d\alpha\, d\lambda$$

$$= \int_{t-4-t_0}^{t-t_0} \int_{\tau-2}^{\tau} x_1(\alpha)\, d\lambda\, d\alpha$$

$$y_1(t-t_0) = \int_{t-4-t_0}^{t-t_0} \int_{\tau-2}^{\tau} x_1(\lambda)\, d\lambda\, d\tau = y_2(t)$$

$\therefore$ The system is time-invariant.

(c) Impulse response of overall system

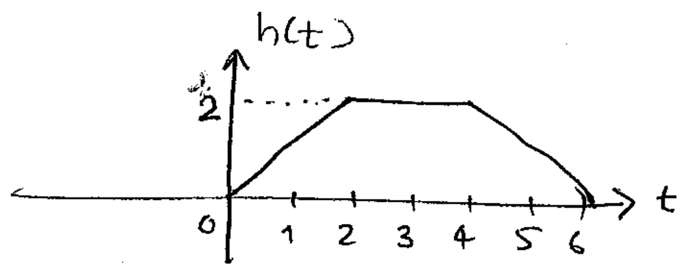
$$h(t) = \int_{t-4}^{t} \int_{\tau-2}^{\tau} \delta(\lambda)\, d\lambda\, d\tau$$

$$= \int_{t-4}^{t} \left[u(\tau) - u(\tau-2) \right] d\tau$$

$$= \int_{t-4}^{t} u(\tau)\, d\tau - \int_{t-4}^{t} u(\tau-2)\, d\tau$$

$$= \begin{cases} 0 & t \leq 0 \\ t & 0 \leq t \leq 4 \\ 4 & t \geq 4 \end{cases} - \begin{cases} 0 & t \leq 2 \\ t-2 & 2 \leq t \leq 6 \\ 4 & t \geq 6 \end{cases}$$

$$h(t) = \begin{cases} 0 & t \leq 0 \\ t & 0 \leq t \leq 2 \\ 2 & 2 \leq t \leq 4 \\ 6-t & 4 \leq t \leq 6 \\ 0 & t \geq 6 \end{cases}$$



(d) causal?

Note that the overall system is LTI

$h(t) = 0$ for $t < 0$

$\therefore$ The LTI overall system is causal.

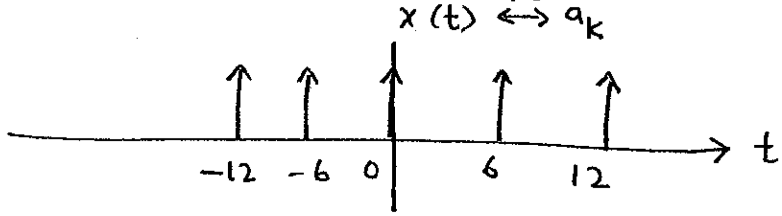
(e) stable?

$$\int_{-\infty}^{\infty} |h(t)|\,dt = 8 < \infty$$

$\therefore$ The system is stable.

(f) $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - k6)$

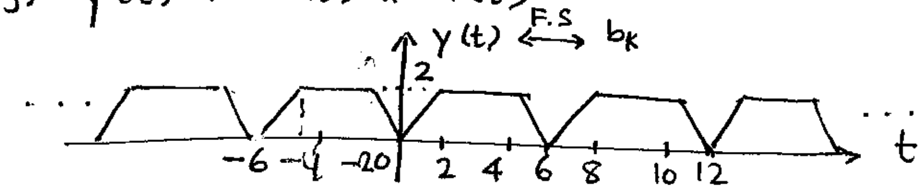
$x(t) \overset{F.S}{\longleftrightarrow} a_k$



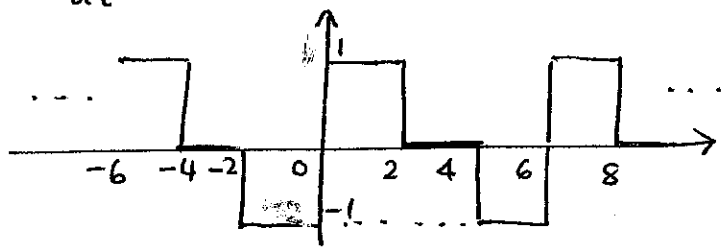
$a_k = \frac{1}{6}$ ←

(g) $y(t) = x(t) * h(t)$

$y(t) \overset{F.S}{\longleftrightarrow} b_k$



$$\frac{d(y(t))}{dt} \overset{F.S}{\longleftrightarrow} m_k$$



$k \neq 0$

$$m_k = \frac{\sin\left(k\frac{2\pi}{6}(1)\right)}{k\pi} e^{-jk\frac{2\pi}{6}(1)} - \frac{\sin\left(k\frac{2\pi}{6}(1)\right)}{k\pi} e^{-jk\frac{2\pi}{6}(-1)}$$

$$= \frac{\sin\left(\frac{k\pi}{3}\right)}{k\pi}\left[e^{-j\frac{k\pi}{3}} - e^{j\frac{k\pi}{3}}\right]$$

$$= \frac{\sin\left(\frac{k\pi}{3}\right)}{k\pi}\left[-2j\sin\left(\frac{k\pi}{3}\right)\right]$$

$$= \frac{2\sin^2\left(\frac{k\pi}{3}\right)}{jk\pi}$$

$$\therefore\ b_k = \frac{m_k}{jk\frac{2\pi}{6}} = \frac{2\sin^2\left(\frac{k\pi}{3}\right)}{jk\pi} \cdot \frac{1}{jk\frac{\cancel{2}\pi}{\cancel{6}\,3}}$$

$$= \frac{-6\sin^2\left(\frac{k\pi}{3}\right)}{k^2\pi^2} \leftarrow \qquad k \neq 0$$

For $k = 0$,

$$b_0 = b_k = \frac{8}{6} = \frac{4}{3} \leftarrow$$

(h) $\omega(t) = y(3t)$

period of $\omega(t) = \frac{6}{3} = 2 \leftarrow$

$c_k = b_k$ (By Time scaling property)

Alternative Solutions for parts (a), (b) and (c) of Problem # 1.

(a) & (b) system 1: $z(t) = \int_{t-2}^{t} x(\tau)\,d\tau$

$$x_1(t) \rightarrow z_1(t) = \int_{t-2}^{t} x_1(\tau)\,d\tau$$

$$x_2(t) \rightarrow z_2(t) = \int_{t-2}^{t} x_2(\tau)\,d\tau$$

$$x_3(t) = ax_1(t) + bx_2(t) \longrightarrow z_3(t) = \int_{t-2}^{t} [a x_1(\tau) + b x_2(\tau)]\,d\tau$$

$$= a \int_{t-2}^{t} x_1(\tau)\,d\tau + b \int_{t-2}^{t} x_2(\tau)\,d\tau$$

$$= a z_1(t) + b z_2(t)$$

$\therefore$ system (1) is linear $\leftarrow$ (A)

system 2:

$$z_1(t) \rightarrow y_1(t) = \int_{t-4}^{t} z_1(\tau)\,d\tau$$

$$z_2(t) \rightarrow y_2(t) = \int_{t-4}^{t} z_2(\tau)\,d\tau$$

$$z_3(t) = a z_1(t) + b z_2(t) \longrightarrow y_3(t) = \int_{t-4}^{t} [a z_1(\tau) + b z_2(\tau)]\,d\tau$$

$$= a \int_{t-4}^{t} z_1(\tau)\,d\tau + b \int_{t-4}^{t} z_2(\tau)\,d\tau$$

$$= a y_1(t) + b y_1(t)$$

$\therefore$ system (2) is linear. $\leftarrow$ (B)

System (1): $x_1(t) \rightarrow z_1(t) = \int_{t-2}^{t} x_1(\tau)\,d\tau$

$$x_2(t) = x_1(t-t_0) \rightarrow z_2(t) = \int_{t-2}^{t} x_2(\tau)\,d\tau = \int_{t-2}^{t} x_1(\tau - t_0)\,d\tau$$

let $\lambda = \tau - t_0$

$$z_2(t) = \int_{t-t_0-2}^{t-2} x_1(\lambda)\,d\lambda$$

$$z_1(t-t_0) = \int_{t-2-t_0}^{t-t_0} x_1(\tau)\,d\tau = z_2(t)$$

$\therefore$ System (1) is TI. $\leftarrow$ (C)

System 2: $z_1(t) \rightarrow y_1(t) = \int_{t-4}^{t} z_1(\tau)d\tau$

$$z_2(t) = z_1(t-t_0) \longrightarrow y_2(t) = \int_{t-4}^{t} z_2(\tau)d\tau$$

$$= \int_{t-4}^{t} z_1(\tau - t_0)d\tau$$

Let $\beta = \tau - t_0$

$$y_2(t) = \int_{t-t_0-4}^{t-t_0} z_1(\beta)\,d\beta$$

$$y_1(t-t_0) = \int_{t-4-t_0}^{t-t_0} z_1(\tau)\,d\tau = y_2(t)$$

$\therefore$ system 2 is TI $\leftarrow$ D

By A & C $\rightarrow$ system 1 is LTI

By B & D $\rightarrow$ system 2 is LTI

By Associative Property of LTI system, the series interconnection of two LTI systems is equivalent to a single LTI system.

$\therefore$ The overall system is linear, & time-invariant.

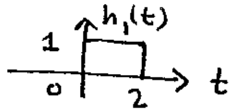
(c) impulse response of overall system

= convolution of individual impulse response of each system in the cascaded connection.

System 1:

$$h_1(t) = \int_{t-2}^{t} \delta(\tau)d\tau = \begin{cases} 1 & 0 < t < 2 \\ 0 & \text{elsewhere} \end{cases}$$

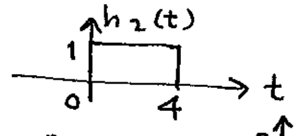
$h_1(t) \Rightarrow$



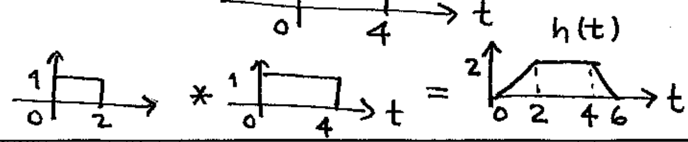
System 2:

$$h_2(t) = \int_{t-4}^{t} \delta(\tau)d\tau = \begin{cases} 1 & 0 < t < 4 \\ 0 & \text{elsewhere} \end{cases}$$

$h_2(t)$



$h(t) = h_1(t) * h_2(t) =$



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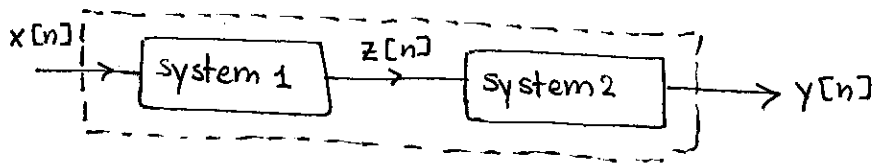


System 1 : $z[n] = x[n-1] + x[n] + x[n+1]$

System 2 : $y[n] = z[n-1] + z[n] + z[n+1]$

Overall system

$$y[n] = z[n-1] + z[n] + z[n+1]$$

$$= x[n-2] + x[n-1] + x[n] + x[n-1] + x[n] + x[n+1] + x[n] + x[n+1] + x[n+2]$$

$$= x[n-2] + 2x[n-1] + 3x[n] + 2x[n+1] + x[n+2]$$

(a) linear?

$x_1[n] \Rightarrow y_1[n] = x_1[n-2] + 2x_1[n-1] + 3x_1[n] + 2x_1[n+1] + x_1[n+2]$

$x_2[n] \rightarrow y_2[n] = x_2[n-2] + 2x_2[n-1] + 3x_2[n] + 2x_2[n+1] + x_2[n+2]$

$x_3[n] = a x_1[n] + b x_2[n] \longrightarrow y_3[n] = a x_1[n-2] + b x_2[n-2] + 2a x_1[n-1] + 2b x_2[n-1] + 3a x_1[n] + 3b x_2[n] + 2a x_1[n+1] + 2b x_2[n+1] + a x_1[n+2] + b x_2[n+2]$

$$= a\left[x_1[n-2] + 2x_1[n-1] + 3x_1[n] + 2x_1[n+1] + x_1[n+2]\right] + b\left[x_2[n-2] + 2x_2[n-1] + 3x_2[n] + 2x_2[n+1] + x_2[n+2]\right]$$

$$= a y_1[n] + b y_2[n]$$

$\therefore$ The overall system is linear.

(b) TI?

$$x_1[n] \Rightarrow y_1[n] = x_1[n-2] + 2x_1[n-1] + 3x_1[n] + 2x_1[n+1] + x_1[n+2]$$

$$x_2[n] = x_1[n-n_0] \rightarrow y_2[n] = x_2[n-2] + 2x_2[n-1] + 3x_2[n] + 2x_2[n+1] + x_2[n+2]$$

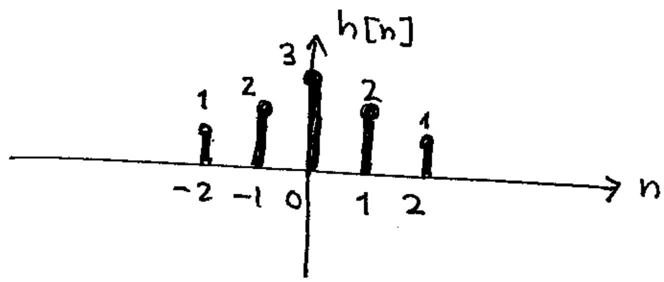
$$= x_1[n-n_0-2] + 2x_1[n-n_0-1] + 3x_1[n-n_0] + 2x_1[n-n_0+1] + x_1[n-n_0+2]$$

$$y_1[n-n_0] = x_1[n-n_0-2] + 2x_1[n-n_0-1] + 3x_1[n-n_0] + 2x_1[n-n_0+1] + x_1[n-n_0+2] = y_2[n]$$

$\therefore$ The system is Time Invariant.

(c) Impulse Response?

$$h[n] = \delta[n-2] + 2\delta[n-1] + 3\delta[n] + 2\delta[n+1] + \delta[n+2]$$



(d) causal?

Note that the overall system is LTI.

$h[n] \neq 0$ for $n < 0$

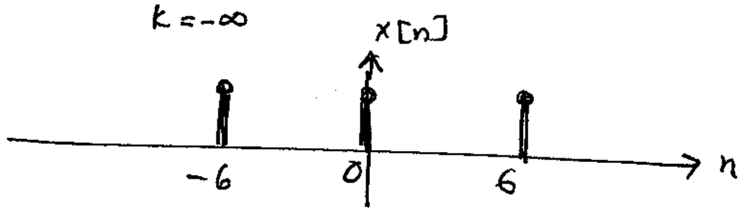
$\therefore$ The system is NOT causal.

(e) stable?

$$\sum_{k=-\infty}^{\infty} |h[n]| = 1+2+3+2+1 = 9 < \infty$$

$\therefore$ The system is stable.

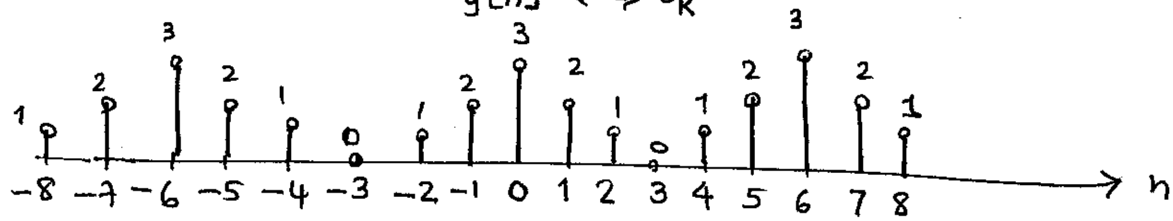
(f) $x[n] = \sum_{k=-\infty}^{\infty} \delta[n - k6]$



$a_k = \frac{1}{6}$ for all k.

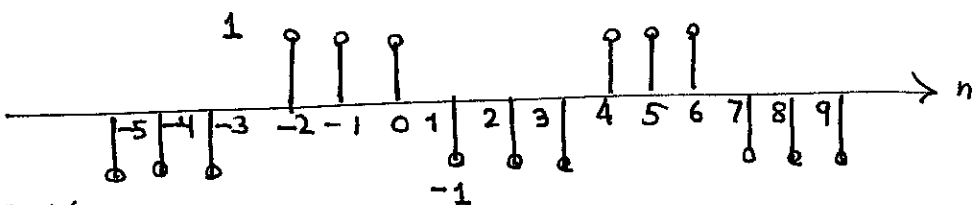
(g) $y[n] = x[n] * h[n]$

$$y[n] \overset{F.S}{\longleftrightarrow} b_k$$



Let $z[n] = y[n] - y[n-1]$

$$z[n] \overset{F.S}{\longleftrightarrow} d_k$$



For $k \neq 0, \pm 6, \pm 12, \ldots$

$$d_k = \frac{1}{6} \frac{\sin\left[2\pi k\left(1+\frac{1}{2}\right)/6\right]}{\sin(\pi k/6)} e^{-jk\frac{2\pi}{6}(-1)} - \frac{1}{6} \frac{\sin\left[2\pi k\left(1+\frac{1}{2}\right)/6\right]}{\sin(\pi k/6)} e^{-jk\frac{2\pi}{6}(2)}$$

$$= \frac{1}{6} \frac{\sin\left(\frac{\pi k}{2}\right)}{\sin\left(\frac{\pi k}{6}\right)} \left[e^{-jk\frac{\pi}{3}} - e^{-j\frac{2k\pi}{3}}\right]$$

$$= \frac{1}{6} \frac{\sin\left(\frac{\pi k}{2}\right)}{\sin\left(\frac{\pi k}{6}\right)} e^{-\frac{jk\pi}{2}} \left[e^{j\frac{\pi}{6}k} - e^{-j\frac{k\pi}{6}}\right]$$

$$= +\frac{1}{\cancel{6}\,3}\,\frac{\sin\left(\frac{\pi k}{2}\right)}{\cancel{\sin\left(\frac{\pi k}{6}\right)}}\,e^{-jk\frac{\pi}{2}}\,(\cancel{2}j)\,\cancel{\sin\left(\frac{k\pi}{6}\right)}$$

$$= +\frac{j}{3}\,e^{-jk\frac{\pi}{2}}\,\sin\left(\frac{\pi k}{2}\right)$$

$$\therefore \quad b_k = \frac{d_k}{\left(1-e^{-jk\frac{2\pi}{6}}\right)}$$

$$= \frac{1}{\left(1-e^{-jk\frac{\pi}{3}}\right)}\,\frac{j}{3}e^{-jk\frac{\pi}{2}}\,\sin\left(\frac{\pi k}{2}\right) \leftarrow$$

for $k \neq 0, \pm 6, \pm 12, \ldots$

for $k = 0, \pm 6, \pm 12$

$$b_k = \frac{9}{6} = \frac{3}{2} \leftarrow$$

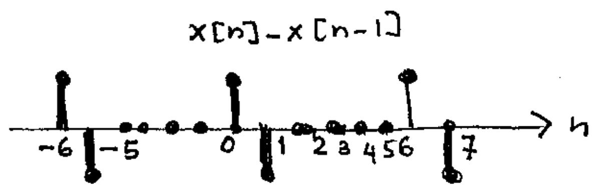
(h) $w[n] = x[n] - x[n-1] \xleftrightarrow{F.S} c_k$

By First difference property,

$$c_k = a_k\left(1-e^{-jk\frac{2\pi}{6}}\right)$$

$$= \frac{1}{6}\left(1-e^{-jk\frac{2\pi}{6}}\right) \leftarrow \quad \text{for all } k.$$

If you visualize $x[n]-x[n-1]$



It is also obvious that

$$c_k = \frac{1}{6} - \frac{1}{6}e^{-jk\frac{2\pi}{6}(1)} \leftarrow \quad \text{for all } k.$$

Alternative Solutions for parts (a), (b) and (c) of problem # 2.

(a) & (b) System 1:

$$z[n] = x[n-1] + x[n] + x[n+1]$$

$$x_1[n] \rightarrow z_1[n] = x_1[n-1] + x_1[n] + x_1[n+1]$$

$$x_2[n] \rightarrow z_2[n] = x_2[n-1] + x_2[n] + x_2[n+1]$$

$$x_3[n] = a\,x_1[n] + b\,x_2[n] \rightarrow z_3[n] = a\,x_1[n-1] + b\,x_2[n-1] + a x_1[n] + b\,x_2[n] + a\,x_1[n+1] + b\,x_2[n+1]$$

$$= a\,[x_1[n] + x_1[n-1] + x_1[n+1]] + b\,[x_2[n-1] + x_2[n] + x_2[n+1]]$$

$$= a\,z_1[n] + b\,z_2[n]$$

$\therefore$ System 1 is linear. $\leftarrow$ (A)

System 2:

$$y[n] = z[n-1] + z[n] + z[n+1]$$

$$z_1[n] \rightarrow y_1[n] = z_1[n-1] + z_1[n] + z_1[n+1]$$

$$z_2[n] \rightarrow y_2[n] = z_2[n-1] + z_2[n] + z_2[n+1]$$

$$z_3[n] = a z_1[n] + b\,z_2[n] \rightarrow y_3[n] = a z_1[n-1] + b\,z_2[n-1] + a z_1[n] + b\,z_2[n] + a\,z_1[n+1] + b\,z_2[n+1]$$

$$= a\,[z_1[n-1] + z_1[n] + z_1[n+1]] + b\,[z_2[n-1] + z_2[n] + z_2[n+1]]$$

$$= a\,y_1[n] + b y_2[n]$$

$\therefore$ System 2 is linear $\leftarrow$ (B)

System 1:

$$x_1[n] \rightarrow z_1[n] = x_1[n-1] + x_1[n] + x_1[n+1]$$

$$x_2[n] = x_1[n-n_0] \rightarrow z_2[n] = x_2[n-1] + x_2[n] + x_2[n+1]$$

$$= x_1[n-n_0-1] + x_1[n-n_0-1] + x_1[n-n_0+1]$$

$$z_1[n-n_0] = x_1[n-n_0-1] + x_1[n-n_0] + x_1[n-n_0+1] = z_2[n]$$

$\therefore$ system 1 is TI. $\longleftarrow$ (C)

System 2:

$$z_1[n] \longrightarrow y_1[n] = z_1[n-1] + z_1[n] + z_1[n+1]$$

$$z_2[n] = z_1[n-n_o] \longrightarrow y_2[n] = z_2[n-1] + z_2[n] + z_2[n+1]$$
$$= z_1[n-n_o-1] + z_1[n-n_o] + z_1[n-n_o+1]$$

$$y_1[n-n_o] = z_1[n-n_o-1] + z_1[n-n_o] + z_1[n-n_o+1] = y_2[n]$$

$\therefore$ System 2 is TI $\longleftarrow$ (D)

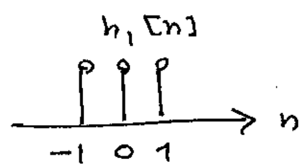
From (A) & (C), System 1 is LTI
(B) & (D), system 2 is LTI

$\therefore$ The overall system is LTI.

(c) impulse response of overall system
= convolution of individual impulse response of each system in the cascaded connection.

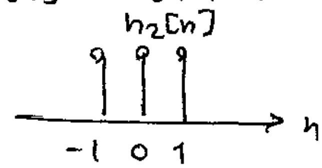
System (1)

$$h_1[n] = \delta[n-1] + \delta[n] + \delta[n+1]$$



System (2)

$$h_2[n] = \delta[n-1] + \delta[n] + \delta[n+1]$$



$$\therefore \quad h[n] = h_1[n] * h_2[n]$$

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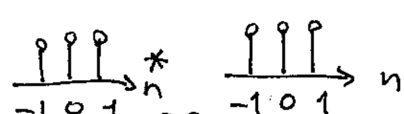

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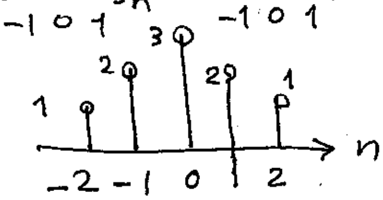

$\longleftarrow$