Convolving with a Dirac Delta Function

$$x(t) * \delta(t) = x(t)$$

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

$$\delta(t-t_1) * \delta(t-t_2) = \delta\left(t-(t_1+t_2)\right)$$

for $-\infty < t_1 < \infty$ and $-\infty < t_2 < \infty$

$$\delta(t+t_0) * \delta(t-t_0) = \delta(t)$$

Properties of Convolution:

i.) Commutative:

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

ii.) Associativity:

$$\left(x_1(t) * x_2(t)\right) * x_3(t) = x_1(t) * \left(x_2(t) * x_3(t)\right)$$

iii.) Distributive:

$$x_1(t) * \left(x_2(t) + x_3(t)\right)$$

$$= x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

Suppose: $x_1(t)$ "turns on" at t_1

$h(t)$ "turns on" at t_2

$$y(t) = x(t) * h(t)$$

$$= x(t) * \delta(t+t_1) * \delta(t-t_1) * h(t) * \delta(t+t_2) * \delta(t-t_2)$$

$$= \left(x(t) * \delta(t+t_1)\right) * \left(h(t) * \delta(t+t_2)\right) * \delta(t-t_1) * \delta(t-t_2)$$

$$= \tilde{x}(t) * \tilde{h}(t) * \delta\left(t-(t_1+t_2)\right)$$

$$= \tilde{y}(t) * \delta\left(t-(t_1+t_2)\right)$$

$$= \tilde{y}\left(t-(t_1+t_2)\right)$$

where: $\tilde{y}(t) = \tilde{x}(t) * \tilde{h}(t)$

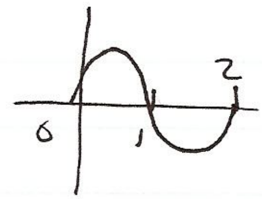
Choose t_1 so $\tilde{x}(t+t_1)$ starts at $t=0$

choose t_2 so $\tilde{h}(t+t_2)$ starts at $t=0$

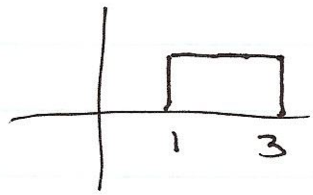
- Then shift result by (t_1+t_2) to the right

Example. Prob. 2.22 (c)

$$x(t) = \sin(\pi t)\ \mathrm{rect}\left(\frac{t-1}{2}\right)$$

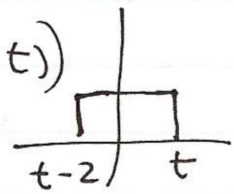


$$h(t) = \mathrm{rect}\left(\frac{t-2}{2}\right)$$



Create $\tilde{h}(t) = h(t+1)$

$$\tilde{y}(t) = \int_0^t \sin(\pi\tau)\,d\tau \quad \tilde{h}(-(\tau-t))$$



for $t-2<0$

$0<t<2$

$$\tilde{y}(t) = -\frac{\cos(\pi\tau)}{\pi}\Big]_0^t$$

$$= -\frac{\cos(\pi t)}{\pi} + \frac{1}{\pi}$$

$$= \frac{1}{\pi}\left\{1 - \cos(\pi t)\right\}$$

$2<t<4$

for $t>2$ (note $x(t)$ and $\tilde{h}(t)$ are same duration)

$$\tilde{y}(t) = -\frac{\cos(\pi\tau)}{\pi}\Big|_{t-2}^{2} = -\frac{\cos(2\pi)}{\pi} + \frac{\cos(\pi(t-2))}{\pi}$$

$$= \frac{1}{\pi}\left\{\cos(\pi(t-2)) - 1\right\}$$

$$y(t) = \tilde{y}(t-1) = \begin{cases} \frac{1}{\pi}\left(1-\cos(\pi(t-1))\right), & 1<t<3 \\ \frac{1}{\pi}\left(\cos(\pi(t-3))-1\right) & 3<t<5 \end{cases}$$