Discrete-Time LTI Systems

- DT Convolution
- Define impulse response of DT System that is both Linear and Time-Invariant (LTI)

\[ \delta[n] \xrightarrow{\text{LTI}} h[n] \]

To easily derive DT convolution formula, we view \( x[n] \) (input) as a sum of amplitude-scaled and time-shifted (Kronecker) Delta functions \( \Rightarrow \) See Fig. 2.1 on pg. 76

\[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \]
- **Time-Invariance dictates:**

\[ \delta[n-h] \rightarrow \text{LTI} \rightarrow h[n-h] \]

- **Homogeneity aspect of linearity dictates:**

\[ x[h] \delta[n-h] \rightarrow \text{LTI} \rightarrow x[h] h[n-h] \]

- **Superposition aspect of linearity dictates:**

\[
\sum_{h=-\infty}^{\infty} x[h] \delta[n-h] \rightarrow \text{LTI} \rightarrow y[n] = \sum_{h=-\infty}^{\infty} x[h] h[n-h] = x[n] * h[n]
\]

See Fig. 2.2 on pg. 79
Summarizing:

\[ x[n] \rightarrow h[n] \rightarrow y[n] = x[n] * h[n] \]

\[ = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \]

There are at least 3 ways to compute DT convolution:

Method 1: collectively sum

Example: \( y[n] = x[n] + x[n-1] + x[n-2] \)

Find output when: \( x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] \)

Impulse response of system?
\( y[n] = h[n] \) when \( x[n] = \delta[n] \)

\( h[n] = \delta[n] + \delta[n-1] + \delta[n-2] \)
\[ x[n] = f[n] + 2 \delta[n-1] + 3 \delta[n-2] \]
\[ y[n] = x[n] \ast h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \]
\[ = x[0] h[n] = h[n] \]
\[ + x[1] h[n-1] = 2 h[n-1] \]
Answer is sum: $y[n]$

$x[n]$ of "length" 3
$h[n]$ of "length" 3

$x[n] * h[n]$ of length $3 + 3 - 1 = 5$

Generally: $x[n]$ of "length" $N_1$
$h[n]$ of "length" $N_2$

$x[n] * h[n]$ of "length" $N_1 + N_2 - 1$
Note: not concerned with initial conditions in this course ⇒ unless stated otherwise assume system is initially at rest ⇒ all initial conditions = 0

Method 2: "run" input signal thru difference equation (Note: all DT LTI systems may be expressed as a difference equation)

- In the previous example: \( x[n] = 0 \) for \( n < 0 \)
  \[ x[0] = 1, \; x[1] = 2, \; x[2] = 3, \; x[n] = 0 \text{ for } n > 2 \]

\( h = 0 \)

\[ y[0] = x[0] + x[1] + x[-2] = 1 + 0 + 0 = 1 \]

\[ y[1] = x[1] + x[0] + x[-1] = 2 + 1 + 0 = 3 \]


\[ y[n] = 0 \text{ for } n > 4 \]
Method 3 Graphical Method similar to that for CT convolution:

\[ y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) \]

1. View \( x(k) \) and \( h[-(k-n)] \) as functions of \( k \)
2. Flip \( h[k] \) about \( k=0 \) to form \( h[-k] \)
3. Time-shift \( h[-k] \) to the right by \( n \) to form \( h[-(k-n)] \)
4. Pointwise-multiply to form product \( x(k)h[-(k-n)] \)
5. Sum the values of the product \( x(k)h[-(k-n)] \) over all \( k \)
6. Ostensibly repeat for each value of \( n \)

See Example 2.3 in text on pg. 83
More generally: \( x[n] = \alpha^n u[n] \)

\[ h[n] = \beta^n u[n] \]

\( \alpha + \beta \)

\[ y[n] = 0 \text{ for } n < 0. \]

For \( n > 0 \):

\[ x[k] = \alpha^k u[k] \]

\[ h[-(k-n)] = \beta^{-(k-n)} u[-(k-n)] \]

\[ y[n] = \sum_{k=0}^{n} \alpha^k \beta^{n-k} = \beta^n \sum_{k=0}^{n} \left( \frac{\alpha}{\beta} \right)^k \]
\[ y(n) = \beta^n \frac{1 - (\frac{\alpha}{\beta})^{n+1}}{1 - \frac{\alpha}{\beta}} = \beta^n \frac{\beta - \frac{\alpha^n}{\beta^n}}{\beta - \alpha} \]

\[ = \begin{cases} \frac{\beta}{\beta - \alpha} \beta^n - \frac{\alpha}{\beta - \alpha} \alpha^n \end{cases} u(n) \]

since starts at \( n = 0 \)

Example 2.4 in text on pg. 85

\[ x(n) = u(n) - u(n-5) = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases} \]

\[ h(n) = \alpha^n \left\{ u(n) - u(n-7) \right\} \]

In contrast to text approach (Method 2), this is short enough to do by Method 1
\( x(0) = x(1) = x(2) = x(3) = x(4) = 1 \)

<table>
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<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x(0)h(n) )</td>
<td>1</td>
<td>( a )</td>
<td>( a^2 )</td>
<td>( a^3 )</td>
<td>( a^4 )</td>
<td>( a^5 )</td>
<td>( a^6 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x(1)h(n-1) )</td>
<td>0</td>
<td>1</td>
<td>( a )</td>
<td>( a^2 )</td>
<td>( a^3 )</td>
<td>( a^4 )</td>
<td>( a^5 )</td>
<td>( a^6 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x(2)h(n-2) )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( a )</td>
<td>( a^2 )</td>
<td>( a^3 )</td>
<td>( a^4 )</td>
<td>( a^5 )</td>
<td>( a^6 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x(3)h(n-3) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( a )</td>
<td>( a^2 )</td>
<td>( a^3 )</td>
<td>( a^4 )</td>
<td>( a^5 )</td>
<td>( a^6 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x(4)h(n-4) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( a )</td>
<td>( a^2 )</td>
<td>( a^3 )</td>
<td>( a^4 )</td>
<td>( a^5 )</td>
<td>( a^6 )</td>
<td>0</td>
</tr>
<tr>
<td>( y(n) )</td>
<td>1</td>
<td>( 1+a )</td>
<td>( 1+a+a^2 )</td>
<td>( 1+a+a^2 )</td>
<td>( 1+a+a^2 )</td>
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</tbody>
</table>

\[ \frac{1-a^3}{1-a} \quad \frac{1-a^4}{1-a} \quad \frac{1-a^5}{1-a} \quad \frac{a^2(1-a^5)}{1-a} \quad \frac{a^4(1-a^3)}{1-a} \quad \frac{a^3(1-a^4)}{1-a} \]
DT convolution satisfies:

1. Commutativity:  \[ x_1(n) * x_2(n) = x_2(n) * x_1(n) \]

2. Associativity:
   \[
   (x_1(n) * x_2(n)) * x_3(n) = x_1(n) * (x_2(n) * x_3(n))
   \]

3. Distributive Property:
   \[
   x_1(n) * (x_2(n) + x_3(n)) = x_1(n) * x_2(n) + x_1(n) * x_3(n)
   \]